The engineer grapples with Theorem 1.1 and Lemma 6.3 of Strang and Fix*

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The best-known statements on error analysis of the finite element method are derived using rigorous mathematical abstractions which are difficult for the average engineer (who remains the biggest user) to grasp. There is much merit in being able to re-derive these using the energy, virtual work and least action principles that the engineer or physicist is more familiar with. In this article, we attempt to do this, to obtain Theorem 1.1 and Lemma 6.3 of Strang and Fix, perhaps the most valuable of all error statements ever made of finite element elastostatics and elastodynamics. We also looked at some interesting atypical problems which arise when errors that appear when finite element discretization is used to solve problems of interest in engineering and applied science are studied. The formal mathematical theorems and lemmas which have been identified in the seminal work of Strang and Fix, are now re-examined for these atypical situations using an engineering approach.

IF a poll were ever to be made to identify the greatest book written on the finite element method, one of the most obvious candidates would be the seminal *Analysis of the Finite Element Method* by Strang and Fix[1]. Although the results from this book have been accessible to the mathematical community working on finite element analysis for three decades now, and has led to a large body of work, the average engineer, who remains the main user of the various finite element codes in engineering practice, still finds it difficult to grapple with the rigorous procedures of functional analysis required to derive the various error statements. Even before Strang and Fix, there was established an excellent tradition of engineering analysis by de Veubeke[2]. He started with a three-field functional for linear elastostatics known as the Hu–Washizu functional[3,4] which allowed displacements, strains and stresses to be varied independently. Recently, Prathap[5,6] enlarged this interpretation to show that if the finite element fields and the exact fields are tracked separately, it is possible to derive a projection theorem relating the approximate stresses to the error in the strains. This can be viewed as an effort to use the energy theorems, the virtual work or least action principles, as the basis for deriving the projection theorems and the energy–error statements that have been handed down to the finite element world by the mathematicians. More recently, Liew and Rajendran[7] showed that instead of starting with the Hu–Washizu functional, the projection theorem can be derived much more simply from the virtual work principle. Here, we examine the motivation for this and re-derive the various error theorems that guide the interpretation of finite element computation of elastostatical and elastodynamical problems, the most famous of them being Theorem 1.1 and Lemma 6.3 from Strang and Fix[1].

The virtual work approach, in a very generous interpretation, can be traced back to Archimedes' derivation of the law of the lever. One can begin to appreciate the grandeur of Archimedes' achievement only if one realizes that Archimedes stated the famous $F_1 l_1 = F_2 l_2$ law, where F_1 is the force applied at a distance l_1 from the fulcrum and F_2 is the force applied at a distance l_2 from the fulcrum on the other arm of the lever, nearly 18 centuries ahead of the invention of the concept of the moment of a force about a point. The way Archimedes went about his derivation, can now be generously interpreted as the very way the finite element method is stated. If the lever arm is virtually displaced (rotated about the fulcrum) through an angle θ, so that the ends of the lever displace by distances d_1 and $-d_2$ respectively, the work done is $W = F_1 d_1 - F_2 d_2$. At the point where equilibrium is achieved, one can invoke the virtual work argument that $W = F_1 d_1 - F_2 d_2 = 0$. This together with the statement of continuity (the lever arm rotates as a rigid body) that the displacements are related to the lengths from the simple Euclidean relationship $d_1/l_1 = d_2/l_2$, leads to the law of equilibrium. Note that the equation of equilibrium has been achieved by invoking only the law of virtual work and the need to ensure continuity.

This is exactly how the finite element procedure is implemented today, twenty-two centuries after Archimedes. The differential equations of equilibrium are considered to be equivalent to an extreme or stationary condition of a functional based on potential energy. The finite element matrices are obtained by setting up this total potential energy and ensuring continuity of the displacements, and a variation of this to establish that where

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the virtual work is zero, we get the condition of equilibrium, and hence the finite element equations for equilibrium. Strang and Fix[1] use such variational principles to study the Ritz and finite element techniques using the rigorous language of functional analysis. It would be tempting now to see if these energy approaches can be extended to make statements about errors originating due to the finite element discretization problem. The most insightful statements available so far are Theorem 1.1 and Lemma 6.3 from Strang and Fix[1]. Our attempt below will be to re-derive this, less rigorously, but with greater mechanical intuition, using the virtual work approach.

Virtual work for elastostatics

Following the nomenclature used in Strang and Fix[1], we write the weak form in terms of the energy inner product for the exact solution u to the problem.

$$a(u, u) = (f, u), \tag{1}$$

$$a(u, u^h) = (f, u^h), \tag{2}$$

where $a(u, u)$ is the bilinear symmetric functional and (f, u) is the integral $\int uf \, dV$-over the system domain V. The first two virtual work statements refer to the exact solution of the elastostatic problem. In eq. (1), the trial function and test function are taken as u and the virtual work argument establishes that eq. (1) is truly satisfied only when u is the exact solution at the point of equilibrium. In eq. (2), we take note of the fact that the test function u^h (the Ritz or finite element solution) need not be the exact displacement function for the virtual work principle to be true. For convenience, we take this to be the discrete finite element displacement field, as long as it is admissible (i.e. satisfies all the geometric boundary conditions).

It is in the next equation that we take the final step towards discretization. Using u^h for both the trial and test functions, we get the actual finite element equations, with the right-hand side leading to the consistent load vector and the left-hand side representing the stiffness matrix.

$$a(u^h, u^h) = (f, u^h). \tag{3}$$

Equation (3) will now reflect the error due to the finite element discretization. We are now in a position to see how the error $e = u - u^h$ can be assessed. Comparing eqs (2) and (3) and noting that the energy inner product is bilinear, we can arrive at

$$a(u, u^h) = a(u^h, u^h),$$

and from this we obtain the projection theorem

$$a(u - u^h, u^h) = 0. \tag{4}$$

The finite element solution is therefore seen to be a best fit or best-approximation solution. In most simple linear elastostatics cases, this would imply that the strains or stresses are obtained in a best fit sense and that there would be points in the element domain where these stresses or strains are accurately computed (superconvergence).

From the fact that the energy inner product is bilinear, we can argue that

$$a(u - u^h, u - u^h) = a(u, u) + a(u^h, u^h) - 2a(u, u^h)$$

$$= a(u, u) - a(u^h, u^h) - 2[a(u, u^h) - a(u^h, u^h)]$$

$$= a(u, u) - a(u^h, u^h) - 2[a(u - u^h, u^h)].$$

Introducing the result from eq. (4), we get an energy–error theorem which can be expressed as

$$a(u - u^h, u - u^h) = a(u, u) - a(u^h, u^h), \tag{5}$$

i.e. energy of the error = error of the energy.

This leads to a useful statement that as the left-hand side of eq. (5) is always positive definite,

$$a(u^h, u^h) < a(u, u). \tag{6}$$

Thus, in a variationally correct approach, the energy inner product of the approximate (Ritz or finite element) solution will always be a lower bound of the exact energy.

Equations (1) to (6) are a restatement of the equations covered by Theorem 1.1 of Strang and Fix[1]. We have started with three virtual work equations (eqs (1) to (3)) and this led easily to a projection theorem (eq. (4)), energy–error theorem (eq. (5)), and a lower bound result (eq. (6)).

Virtual work for elastodynamics

From the foregoing, it is a simple matter to extend these results to a problem in elastodynamics. Now, the kinetic energy of motion enters into the picture, and a classical variational basis for this is already available through the Lagrangian or Hamiltonian statements, where both potential energy and kinetic energy enter into the functional.

Unlike Strang and Fix[1], where the development of the argument is based on the Rayleigh quotient, we shall write the weak form in terms of the energy inner product for the exact solution u to the problem and replace the loading term f with the inertial force term $\omega^2 \rho u$. The earlier equations yield

$$a(u, u) = \omega^2 \cdot (\rho u, u), \tag{7}$$

$$a(u, u^h) = \omega^2 \cdot (\rho u, u^h), \tag{8}$$

where ρ is the inertia density of the domain. Again, the first two virtual work statements refer to the exact solution of the elastodynamic problem. In eq. (7), the trial function and test function are taken as u and the virtual work argument establishes that eq. (7) is truly satisfied only when u is the exact eigenfunction and ω^2 is the exact eigenvalue. In eq. (8), we take note of the fact that the

test function u^h (the Ritz or finite element solution) need not be the exact displacement function for the virtual work principle to be true, while ω^2 remains the exact eigenvalue. Again, we take u^h to be the discrete finite element displacement field, as long as it is admissible (i.e. satisfies all the geometric boundary conditions).

We now take the step towards discretization. Using u^h for both the trial and test functions, we get the actual finite element equations, with the right-hand side leading to the consistent mass matrix and the left-hand side leading to the stiffness matrix.

$$a(u^h, u^h) = (\omega^h)^2 \cdot (\rho u^h, u^h), \tag{9}$$

This equation will now reflect the error due to the finite element discretization, appearing both in the eigenfunction and in the eigenvalue. This makes the assessment of the errors a trifle more complicated than in the elastostatics case earlier, as there is the error in the eigenfunction, $u - u^h$ as well as the error in the eigenvalue, $(\omega^h)^2 - \omega^2$ to be assessed. Comparing eqs (8) and (9) and also noting that the energy inner product is bilinear, we can arrive at

$$[a(u, u^h) - a(u^h, u^h)] - [\omega^2 \cdot (\rho u, u^h) - (\omega^h)^2 \cdot (\rho u^h, u^h)] = 0,$$

or

$$a(u - u^h, u^h) - (\omega^2 \rho u - (\omega^h)^2 \rho u^h, u^h) = 0 \tag{10}$$

becomes the projection theorem for elastodynamics.

The finite element solution is still seen to be a best fit or best-approximation solution. However, unlike the simple linear elastostatics cases, this would imply that the strains or stresses are only nearly obtained in a best fit sense. It is no longer possible now to relate the error of the energy to the energy of the error, as was possible for the elastostatics case as in eq. (5) earlier. We shall first take up the error of the energies. From eqs (7) and (9), we have,

$$[a(u, u) - a(u^h, u^h)] - [\omega^2 \cdot (\rho u, u) - (\omega^h)^2 \cdot (\rho u^h, u^h)] = 0. \tag{11}$$

We now take up the case of the energy of the errors. For this, we have

$$\begin{aligned} a(u - u^h, u - u^h) &= a(u, \rho u) + a(u^h, u^h) - 2a(u, u^h) \\ &= \omega^2 \cdot (\rho u, u) - 2\omega^2 \cdot (\rho u, u^h) + (\omega^h)^2 \cdot (\rho u^h, u^h) \\ &= \omega^2 \cdot (\rho u, u) - 2\omega^2 \cdot (\rho u, u^h) + (\omega)^2 \cdot (\rho u^h, u^h) \\ &\quad + [(\omega^h)^2 - \omega^2] \cdot (\rho u^h, u^h) \\ &= \omega^2 \cdot (\rho u - \rho u^h, u - u^h) + [(\omega^h)^2 - \omega^2] \cdot (\rho u^h, u^h). \end{aligned} \tag{12}$$

This is indeed Lemma 6.3 of Strang and Fix[1]. We started with three virtual work equations (eqs (7) to (9)) and this led easily to a projection theorem (eq. (10)), and separate energy–error theorem (eq. (11)) and error–energy theorem (eq. (12)). Unlike the case for elastostatics, we cannot derive a simple relationship between the energy of the error and the error of the energy. Also, it is not easy to show that for a conforming and variationally correct formulation, the discretized eigenvalue would always be higher than the exact eigenvalue, whereas for the elastostatics problem, we did have an elegant lower bound result (eq. (6)). This is because an eigenvalue problem does not give a unique displacement field u or u^h. One can side-step this issue by introducing into eq. (11), the idea of a normalized generalized mass, where $(\rho u, u) = (\rho u^h, u^h) = 1$. This gives us,

$$(\omega^h)^2 - \omega^2 = -[a(u, u) - a(u^h, u^h)].$$

What is important to us now is that the error is still governed by the error in the strain energy, and therefore, as long as no variational crimes are committed, the orders of convergence for the elastostatic case should apply.

Locking and the best-approximation nature of fem solutions

Locking is a pathological situation where finite elements with multiple strain components often display spurious stress oscillations and very high spurious stiffness[5,8–11]. These come in various manisfestations, e.g. shear locking in Timoshenko beam elements and Mindlin plate elements, membrane locking in arch elements, parasitic shear in plane stress elements, etc.

Recently, Mukherjee and Prathap[12,13] have extended the field-consistency interpretation[5] using the function space approach and the argument that a finite element solution is a best fit projection to explain the origin of the locking phenomena[1]. Consider the strain–displacement matrix $[B]$ which connects the element strain vector $\{\varepsilon^{he}\}$ to the element nodal displacement $\{\delta^e\}$ as

$$\{\varepsilon^{he}\} = [B]\{\delta^e\}. \tag{13}$$

We define here the stiffness inner product in an element e as $\langle x, y\rangle = \int \{x\}^T [D] \{y\}\, dV$, where $[D]$ is the element rigidity matrix. At the element level, the bilinear function can be expressed as twice the element strain energy, or the norm squared of the element strain vector

$$\begin{aligned} a(u^h, u^h)^e &= \|\varepsilon^{he}\|^2 = \langle \varepsilon^{he}, \varepsilon^{he} \rangle \\ &= \{\delta^e\}^T \int_e [B]^T [D][B]\, dV \{\delta^e\}. \end{aligned} \tag{14}$$

From eq. (3), at an element level,

$$a(u^h, u^h)^e = a(u, u^h)^e,$$

or

$$\|\varepsilon^{he}\|^2 = \langle \varepsilon^{he}, \varepsilon^e \rangle,$$

or

$$\{\delta^e\}^T \int_e [B]^T [D][B]\, dV \{\delta^e\} = \{\delta^e\}^T \int_e [B]^T [D] \{\varepsilon^e\}\, dV,$$

where $[D]$ is the element rigidity matrix. For arbitrary $\{\delta^e\}$, one gets the *normal equation* as

$$\int_e [B]^T [D][B]\, dV\, \{\delta^e\} = \int_e [B]^T [D]\{\varepsilon^e\}\, dV. \quad (15)$$

From eq. (15), it is easy to perceive that finite element strain vectors in an element $\{\varepsilon^{he}\}$ are the best fit in the least square error sense, i.e. they are the orthogonal projections of the analytical strain vector $\{\varepsilon^e\}$ onto a function subspace B, that originates from the element strain–displacement matrix B. In fact, one may determine the m numbers of the orthogonal basis vectors $\{v_i\}$ spanning the m-dimensional function subspace B ($\langle v_i, v_j\rangle = 0$, $i \neq j$) so that the best fit strain vector in the element can be obtained as the orthogonal projection from the formula

$$\{\varepsilon^{he}\} = \sum_{j=1}^{m} \frac{\langle v_j, \varepsilon^e\rangle}{\langle v_j, v_j\rangle}\{v_j\} \quad \langle v_i, v_j\rangle = 0 \text{ for } i \neq j. \quad (16)$$

A subspace B for orthogonal projection is field-inconsistent when it *cannot* be spanned by *standard* orthogonal basis vectors[12,13]. *Locking occurs when the finite element formulations effectively project the analytical strain vector onto such a field-inconsistent subspace resulting from a field-inconsistent strain–displacement relationship*. The approximate (i.e. locked) element strains, obtained as components of a field-inconsistent projection, show spurious stiffness properties and strain oscillations. In shear locking in the Timoshenko beam element, this is shown as reduced bending strains but violent shear-strain oscillations[12,13].

Despite the significant errors and the delay in convergence that is produced in locking situations, it can be shown that the energy–error rule in eq. (5) is still satisfied[12,13], since the locked strain vector in an element, with its oscillating and diminished strain components, is still the orthogonal projection of the analytical strain vector onto an inconsistent subspace.

It was also shown[12,13] that *reduced integration* can be employed to eliminate locking by replacing the original field-inconsistent matrix $[B]$ by a lower order field-consistent matrix $[B^*]$, thereby eliminating spurious strain oscillations and stiffness enhancement arising from field inconsistency. Under such a situation, and provided no variational incorrectness is generated through reduced integration, eq. (15) can be replaced with

$$\int_e [B^*]^T [D][B^*]\, dV\, \{\delta^{e*}\} = \int_e [B^*]^T [D]\{\varepsilon^e\}\, dx, \quad (17)$$

where the normal equation is generated with respect to the new field-consistent subspace B^*. The element strains can be then determined from the projection formula in eq. (16) with the orthogonal basis vectors as the *standard* basis vectors that span the subspace B^*.

Lumped mass elastodynamics and the best approximation nature

The weak form of the elastodynamic differential equation yields a mass matrix $[M^{ce}]$ for an element e from the inner product with the approximate modal function in the following manner.

$$\begin{aligned}(\rho u^h, u^h) &= \sum_e \int_e \{u^h\}^T \rho \{u^h\}\, dV \\ &= \sum_e \{\delta^e\}^T \int_e [N]^T \rho [N]\, dV\, \{\delta^e\} \\ &= \sum_e \{\delta^e\}^T [M^{ce}]\{\delta^e\}, \end{aligned} \quad (18)$$

where $[N]$ is the shape function matrix for the approximate modal displacement function $\{u^h\} = [N]\{\delta^e\}$. The *consistent* mass matrix for an element is given by

$$[M^{ce}] = \int_e [N]^T \rho [N]\, dV. \quad (19)$$

If the mass matrix is computed according to eq. (19), then the elastodynamic error eqs (10) to (12) are satisfied, since it is consistent with the weak form of the elastodynamic differential equation.

However, for computational convenience, engineers often use the *lumped* mass matrix instead of the consistent one. In a lumped mass matrix, all the off-diagonal elements are equal to zero, and the masses are lumped only in the diagonal elements of the matrix. With the lumped mass matrix, the inner product of eq. (18) is replaced by the expression

$$\sum_e \{\delta^{e*}\}^T [M^{le}]\{\delta^{e*}\} = \sum_e \sum_i m_i^e (\delta_i^{e*})^2, \quad (20)$$

where m_i^e is the mass associated with the ith diagonal term for the lumped mass matrix $[M^{le}]$ for the element e. Using the lumped mass formulation effectively replaces eq. (9) by

$$a(u^{h*}, u^{h*}) = (\omega^{h*})^2 \cdot \sum_e \sum_i m_i^e (\delta_i^{e*})^2,$$

or

$$\sum_e a(u^{h*}, u^{h*})^e = (\omega^{h*})^2 \cdot \sum_e \sum_i m_i^e (\delta_i^{e*})^2, \quad (21)$$

where the approximate modal displacement function for the lumped mass in an element is given by $\{u^{h*}\} = [N]\{\delta^{e*}\}$ and the corresponding eigenvalue is $(\omega^{h*})^2$. If $\{u^h\}$ and $\{\omega^h\}^2$ are replaced by $\{u^{h*}\}$ and $(\omega^{h*})^2$, then eqs (10) and (12) are violated. In other words, *elastodynamic results of finite element analysis with the lumped mass matrices are variationally incorrect*, with the lumping process having disturbed the total kinetic energy of the system.

A variationally correct and conforming finite element formulation for elastodynamics with consistent mass matrix

always yields eigenvalues (natural frequencies) higher than that obtained by analytical methods for arbitrary meshing. This is not necessarily true if lumped mass formulation is employed. In fact, lumped mass analysis yields eigenvalues which may be lower or higher than the consistent mass frequencies, but also either lower or higher than or even equal to the exact eigenvalue, depending on the position of the nodes[14].

Non-conforming elements and the energy–error rules

In the foregoing treatment of the use of virtual work and energy theorems to derive the projection theorems and the energy–error rules, we have implicitly assumed that the trial and test functions are admissible ones. Thus, in a finite element approach, not only are the essential boundary conditions expected to be satisfied exactly, it is also expected that across element edges, the continuity conditions are exactly satisfied. Elements which do not fulfil this are called non-conforming elements. We can expect that in such formulations, the theorems we have derived so far will be disturbed, and that the assurance on the upper-bound nature described by eq. (6) for elastostatics, or as described by eq. (12) for elastodynamics, will be lost. Thus, our studies with finite element models of plate vibration using the BFS (conforming) and ACM (non-conforming) elements showed that while the former always produced frequencies which were higher than the exact frequencies, the latter produced frequencies which could be higher or lower than the exact frequencies[15]. In other words, boundedness was lost because of the non-conforming nature of the formulation of the ACM element.

Some situations where the best fit paradigm in an element is violated

Next it will be shown that at an *element* level, the best fit paradigm is valid only when the normal eq. (16), that follows from eq. (15), is satisfied. At a global level, boundary terms in the weak form vanish due to either zero forces or vanishing of the test function from Dirichlet conditions. Therefore, they do not appear in the weak forms for the global system in eqs (1) and (2). However, at an element level, the boundary terms do not necessarily vanish, since vanishing Dirichlet conditions do not necessarily apply at the element nodes, and there can be nodal reactions from adjacent elements or supports. Including the work done due to the displacements at the element boundary, one can express the weak forms at an element level as

$$a(u, u)^e = (f, u)^e + [uR^e]_b, \tag{22}$$

and

$$a(u, u^h)^e = (f, u^h)^e + [u^h R^e]_b, \tag{23}$$

where R^e represents the analytical nodal reaction forces acting on the element from adjacent elements/supports. The terms $[uR^e]_b$ and $[u^hR^e]_b$ represent respectively, the work done by these nodal forces on the exact and test functions u and u^h in the element.

If the exact function u of eq. (23) is now replaced by the finite element test function u^h in an element e of the discretized system, the equilibrium equation at an element level is given by

$$a(u^h, u^h)^e = (f, u^h)^e + [u^h R^{he}]_b, \tag{24}$$

where now the nodal reaction force at the element level due to approximation is given by R^{he} which is necessary for the equilibrium of an element having an approximate test function u^h as its displacement. Note that while the weak forms in eqs (22) and (23) are developed from the exact differential equation with zero residual, eq. (24) is developed on the basis of a vanishing integral of the weighted residual (non-zero) with the test function. Subtracting eq. (23) from eq. (24), one obtains

$$a(u^h, u^h)^e - a(u, u^h)^e = [u^h(R^{he} - R^e)]_b. \tag{25}$$

Thus it is evident that only for problems in which the discretization process in finite element analysis conserves the nodal reactions (i.e. $R^{he} = R^e$), the right-hand side of eq. (25) vanishes in all the elements. Thus the best fit paradigm, based on the validity of eqs (15) and (16), tacitly assumes that finite element approximations do not disturb the analytical values of the nodal reactions in any element due to its connectivity to adjacent elements/supports. In other words, the best fit paradigm for finite element analysis remains valid only if no spurious nodal forces are excited due to the inherent approximation from the discretization process.

There are cases where the best fit paradigm and therefore eqs (15) to (17) are violated by finite element results, since the right-hand side of eq. (25) does not vanish. One example is the finite element solution of the non-eigenvalue Sturm–Liouville-type differential equation for a bar with axial loading, given by

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) + k^2 u = f.$$

The bilinear functionals at the element level can be expressed in terms of the stiffness inner products

$$a(u^h, u^h)^e = \| \varepsilon^{he} \|^2 = \langle \varepsilon^{he}, \varepsilon^{he} \rangle$$

and

$$a(u, u^h)^e = \langle \varepsilon^e, \varepsilon^{he} \rangle.$$

Here we need to define the extended approximate and exact strain vectors and the rigidity matrix for an element as:

$$\{\varepsilon^{he}\} = \{du^h/dx, u^h\}^T, \{\varepsilon^e\} = \{du/dx, u\}^T,$$

$$[D] = \begin{bmatrix} EA & 0 \\ 0 & k^2 \end{bmatrix}.$$

The term k^2 in the equation introduces errors in the nodal reactions in the element, and therefore it effectively engenders spurious nodal forces from the finite element discretization procedure. Thus the finite element result at an element level will not be the best fit as given by eqs (15) to (17). It will deviate from the one that is the best fit given by eq. (17), by an amount exactly equal to the response due to these spurious nodal forces. This effect is eliminated completely when $k^2 = 0$. An analysis confirming this violation of the best-approximation nature in a Sturm–Liouville and a boundary value Dirichlet problem is reported in Sangeeta *et al.*[16].

In this article, we have looked at some interesting atypical problems which arise when errors that appear when finite element discretization is used to solve problems of interest in engineering and applied science, are studied. Formal mathematical theorems and lemmas which have been identified in the seminal work of Strang and Fix[1] have been re-examined using an engineering approach for such atypical situations.

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