Consider the case when $r_2(n)$ is not equal to 1. Since the question regarding $r(n)$ being even for all $n \geq n_0$ seems difficult, let us ask (a hopefully simpler) question.

If A is a infinite subset of such that $r_2(n)$ is different from 1 for all $n \geq n_0$, how small can A be?

Let us start with an example.

Let $A = \{2^k + 2^l : k,\ l \in \ \cup \{0\}\}$, then $r_2(n)$ is different from 1 for all $n \geq 10$ and $|A(x)|$ is around $\frac{1}{2}(\ln x)^2$.

Nicolas *et al.*[8] proved the following theorem.

Theorem 13. If A is an infinite subset of such that $r(A, n) \neq 1$ for all sufficiently large natural numbers n, then

$$\lim\ \sup |A(x)| \left(\frac{\ln \ln x}{\ln x}\right)^{3/2} \geq \frac{1}{20}.$$

The authors of this article have proved the following result.

Theorem 14. (Ref. 9) There exists an absolute constant $c > 0$ with the following property: for any infinite subset A of such that $r(A, n) \neq 1$ for all sufficiently large natural numbers n,

$$|A(x)| \geq c\left(\frac{\ln x}{\ln \ln x}\right)^2 \text{ for all } x \text{ sufficiently large.}$$

This shows that the example which we discussed here is essentially best possible.

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Fast and efficient algorithms for solving ordinary differential equations through computer algebra system

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Ordinary differential equations (ODE) occur in several branches of science and technology. One example may be study of particle interaction in electrorheological (ER) fluids. These are fluids whose properties change when they are exposed to an electric field. These fluids have important applications in many fields, automotive industries in particular. Calculations of interactions between particles suspended in fluid are carried out through the solution of ODE with regular singular point. Series solutions to such problems provide highly accurate results if a large number of terms in the series expansion are included. Based on this, several routines, to be used as a package in Computer Algebra System Maple®, are developed to solve the linear homogeneous ordinary differential equations with a regular singular point. These fast and efficient algorithms show significant improvements over existing routines in terms of memory and computational time requirements. The present algorithms provide the correct answer for many differential equations much more efficiently. Using these tools, a large number of terms in the series expansions can be included to get highly accurate solutions of ordinary differential equations.

COMPUTER Algebra Systems (CAS) are simply the programs which enable one to manipulate mathematical expressions symbolically. One of the biggest attractions of CAS is their ability to manipulate long expressions. For most computer literates, the word *computing* means number crunching or numerical calculations. Manipulation of complex mathematical expressions is considered a daunting task for computers. Before computers appeared on the scene, a calculation usually consisted of a mixture of numerical calculation and calculation by mathematical formulas or algebraic calculation. All the numerical calculations were preceded by a manipulation of algebraic formulas, if the work was to be within the bounds of what is humanly possible. In the 19th century, several large calculations have a substantial number of formula manipulations. Among the famous calculations was Le Verrier's calculation of the orbit of Neptune, which started from the disturbances of the orbit of Uranus, and led to the discovery of Neptune. The most impressive and probably the largest calculation with pencil and paper is by the French astronomer Charles Delaunay[1]. He took 10 years to calculate the orbit of the moon as a function of

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time, and another 10 years to check it. He would need only about 10 min if he had a computer with algebraic manipulation capabilities like Maple®.

It is for such large and complicated algebraic expressions, scientists developed such systems which can share the burden of doing algebra in simplifying the mathematical expressions. Computer algebra can save both time and effort in solving a wide range of problems. In general, the solution obtained through CAS is much more accurate than the traditional techniques. A major problem in the manipulation of complicated algebraic expressions is intermediate expression swell. CAS are well equipped to handle such problems through thoughtful programming[2]. A lot of problems, which were stopped in the past due to their huge size, can be solved using these systems. Algebraic solutions are always exact as opposed to their numerical counterparts. Errors arising from rounding and truncation of numerical quantities in the intermediate steps make the final solution approximate and sometimes unacceptable. On the other hand, the accuracy in CAS is controlled by software; therefore the user has total command over level of accuracy needed. Several applications of computer algebra systems were illustrated in refs 2, 3.

Ordinary differential equations with ordinary or singular point occur extensively in several branches of sciences. The exact solution of these equations is one of the most important requirements in many scientific applications. One of the current applications is in studying the behaviour of electrorheological (ER) fluids. ER fluids are made by suspending electrically susceptible particles in fluid. When an external electric field is applied to the system, the viscosity of the ER fluid can change by an order of magnitude. These fluids are being studied in particular by the automotive industry with a view to creating new vibration-control devices, clutch mechanisms, etc. The research in this field has diversified in two directions, studies of the properties of ER fluids by the experimentalists in order to manufacture a suitable ER fluid, and theoretical studies by applied mathematicians and physicists. The theoretical studies, which at this time are still in their early stages, are to investigate systems, such as suspensions, that show strong departures from the simple Newtonian laws of fluid flow. The first calculation of the effective flow properties of a suspension in terms of its constituents is usually taken to be Einstein's[4]. For a comprehensive review of this topic, the reader is referred to Batchelor[5] and Jeffrey and Acrivos[6].

Latta and Hess[7] studied the problem of incompressible potential flow past a sphere in contact with a plane. The velocity at large distance is taken to be uniform and parallel to the plane. This problem is a simple example where we can exploit CAS in order to solve a second order ordinary differential equation with a singular point at the origin.

Due to the singular point in the differential equation, it is usually solved numerically by using a series expansion about the singular point to get the first steps of the numerical solution.

Many results for lubrication theory problems can be obtained by considering only the flow in the gap between the close surfaces. Other results require a solution for the flow outside the gap. Attempts to include the effects from outside the gap sometimes lead to unusual problems in the solution of ordinary differential equations. An example of such a problem is the calculation of the couple acting on a sphere of radius a and the couple acting on a plane wall, in the case in which the sphere moves parallel to the wall, a distance h from it. O'Neill and Stewartson[8] studied this problem using matched inner and outer expansions, and left unsolved some questions connected with the numerical constants they obtained.

Latta and Hess[7] and O'Neill and Stewartson[8] used numerical methods for solving such problems, but these methods do not provide adequately accurate solutions. Series solution is another method for solving such equations. The numerical solution can be replaced with a large number of terms in the series solution. By expanding this series solution along the axis and matching it to an asymptotic solution, we obtain a solution that is more accurate than a numerical one, with less effort. It is only possible if efficient algorithms and programs are available in computer algebra systems. In order to fulfil these requirements, we developed a fast and efficient algorithm for series solution of the ordinary differential equations for the computer algebra system Maple®. The older version of the software used simple textbook treatment for the topic. Programming errors and simplified treatment gives several errors in the solution and memory and time requirement are enormous. The old version of Maple used the conventional textbook approach to find the coefficients. But they are not well suited for research problems. In this study, the recurrence relations are directly obtained, which has resulted in the improvement of time and efficiency, as shown in the text that follows.

In view of the above considerations, we can start with our first design decision: in which form do we return our solutions? The aim of the package is to address differential equations arising in research problems. Therefore we have decided to return only explicit series, calculated to as many terms as requested (within memory limits). It is assumed that the user will present an equation (or set of equations) containing coefficient functions that will need to be expanded, and that the user will ask for many terms of the series solutions. Because the second-order case is so common, it makes sense to include a special code for that case whenever there is a useful gain in efficiency. Another special case that should be considered separately is that of a linear equation compared with a nonlinear one. Again, the gains in efficiency and the common occurrence of linear systems argue for special code.

A final consideration in the design of the package, one that every package must be aware of, is that returning no solution is better than returning a wrong solution. Therefore it is important to make clear, both in the theorems on which the programming is based, and in the programs themselves, exactly what class of equations can be solved by the current method.

The present program, *dsolve2*, expects as input a linear ordinary differential equation and an expansion point about which the solution is desired. A brief review of linear differential systems and the classification of the expansion point are given.

The general linear nonhomogeneous ordinary differential equation is of the form

$$p_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}y(x)+p_{n-1}(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}y(x)+\cdots+ p_0(x)y(x)=f(x), \quad (1)$$

which may symbolically be written as

$$L(y)\equiv\{p_n\partial^n+p_{n-1}\partial^{n-1}+\cdots+p_1\partial+p_0\}y=f(x),$$

where L is a linear differential operator of order n. The coefficients p_0, p_1, p_2, ..., p_n and the nonhomogeneous term $f(x)$ are continuous, single-valued functions of x, defined on an interval $[a, b]$.

The linear differential eq. (1) together with boundary conditions on $y(x)$ or its derivatives forms a linear differential system. However, usually we shall have to solve eq. (1) without boundary conditions, requiring the definition of a general solution. The general solution of eq. (1) can be written as

$$y=y_c+y_p,$$

where y_c is the complimentary function and y_p is the particular integral.

Let the origin be a regular singular point of the second order homogeneous linear differential equation $Ly = 0$. Rewrite $Ly = 0$ as

$$\sum_{k=0}^{2}x^{2-k}Q_k\partial^{2-k}y(x)=0, \quad (2)$$

where

$$Q_0(x)=1,$$

$$Q_k(x)=x^k\frac{p_{2-k}(x)}{p_2(x)},\quad k=1,2.$$

Since the origin is a regular singular point, $Q_1(x)$ and $Q_2(x)$ are analytic in origin. Let

$$Q_k(x)=\sum_{i\geq 0}q_i^{(k)}x^i, \quad (3)$$

and let $y_1(x)$ and $y_2(x)$ be two fundamental Frobenius series solutions related to the origin. Then the first solution $y_1(x)$ corresponding to the larger root r_1 of the indicial equation,

$$F(r)=r(r-1)+q_0^{(1)}+q_0^{(2)}=0, \quad (4)$$

correct to $O(x^{D+1})$ is given by

$$y_1(x)=\sum_{n=0}^{D}y_n^{(1)}x^{n+r_1},\quad y_0^{(1)}\neq 0, \quad (5)$$

where

$$y_n^{(1)}=\frac{-\sum_{j=0}^{n-1}[(r_1+j)q_{n-j}^{(1)}+q_{n-j}^{(2)}]y_j^{(1)}}{F(r_1+n)}\quad n\geq 1.$$

The second linearly independent Frobenius series solution $y_2(x)$ corresponding to the second root r_2 of the indicial equation can be found as follows, ($r_1 - r_2 \neq 0$ and is not integer):

The second solution $y_2(x)$ correct to $O(x^{D+1})$ is then given by

$$y_2(x)=\sum_{n=0}^{D}y_n^{(2)}x^{n+r_2},\quad y_0^{(2)}\neq 0, \quad (6)$$

where

$$y_n^{(2)}=\frac{-\sum_{j=0}^{n-1}[(r_2+j)q_{n-j}^{(1)}+q_{n-j}^{(2)}]y_j^{(2)}}{F(r_2+n)},\quad n\geq 1.$$

When $r_1 - r_2 = 0$, the second linearly independent solution $y_2(x)$ correct to $O(x^{D+1})$ is given by

$$y_2(x)=y_1(x)\ln(x)+\sum_{n=1}^{D}y_n^*x^{n+r_1}, \quad (7)$$

where

$$y_1^*=\frac{-2y_1^{(1)}-q_1^{(1)}y_0^{(1)}}{F(r_1+1)},$$

and

$$y_n^*=\frac{-\left[2ny_n^{(1)}+\sum_{j=1}^{n}q_j^{(1)}y_{n-j}^{(1)}+\sum_{j=1}^{n-1}\{(r_1+j)q_{n-j}^{(1)}+q_{n-j}^{(2)}\}y_j^*\right]}{F(r_1+n)},$$

$$n\geq 2.$$

When $r_1 - r_2 = N$, a positive integer, the second Frobenius series solution $y_2(x)$ correct to $O(x^{D+1})$ is given by

$$y_2(x) = K y_1(x) \ln x + \sum_{n=0}^{D} y_n^* x^{n+r_2}, \qquad (8)$$

where

$$y_n^* = \frac{-\sum_{j=0}^{n-1} [(r_2 + j) q_{n-j}^{(1)} + q_{n-j}^{(2)}] y_j^*}{F(r_2 + n)}, \quad 1 \le n \le N-1$$

$$= \frac{-\sum_{j=0}^{K-1} [(r_2 + j) q_{N-j}^{(1)} + q_{N-j}^{(2)}] y_j^*}{N},$$

$$y_n^* = \frac{\left[K \left\{ (2n-N) y_{n-N}^{(1)} + \sum_{j=1}^{n-N} q_j^{(1)} y_{n-N-j}^{(1)} \right\} - \sum_{j=0}^{n-1} [(r_2 + j) q_{n-j}^{(1)} + q_{n-j}^{(2)}] y_j^* \right]}{F(r_2 + n)}, \quad n \ge N+1.$$

The coefficient of y_N^* is $F(r_1)$, which is zero.

Several routines are developed using the above recurrence relations to find the series solution of linear ODEs. The program is divided into several small routines to carry out specific tasks. The efficiency of the programs was improved with the help of a number of simplifications used in intermediate steps. In computer algebra systems, these improvements are of utmost importance. The speed and efficiency of the programs allow users to obtain a large number of terms in the series expansion which results in highly accurate solutions. The results show a tremendous improvement, in terms of speed and memory requirements, over the textbook approach. As an example, a comparison of the CPU time and memory requirement for the present algorithms is made with the textbook approach and the results are shown in Table 1. All processing is carried out on a PC.

It is evident that for higher number of terms considered for the solution, the conventional approach for coefficient determination is highly inefficient compared to the present proposal, both in speed and memory.

Table 1. Comparison of speed and efficiency between present routines and standard textbook approach, when roots of the indicial equation are different

	CPU time (seconds)		Words	
Order	Standard	Present	Standard	Present
25	1298	3	720,764	361,673
30	5228	6	1,048,384	372,238
35	19,194	9	1,326,861	372,238
45	1,59,565	15	2,391,626	394,432
75	–	48	–	396,263
100	–	92	–	604,734

Ordinary differential equations frequently occur in several branches of mathematical sciences and their applications are many. Numerical solutions to these equations are often estimated but may not be sufficiently accurate in some applications. For example, Latta and Hess[7] and O'Neill and Stewartson[8] could not get the desired accuracy in their results using numerical methods. Series solution may provide an answer to such problems, but to achieve highly accurate results, a large number of terms in the series are required. Fast and efficient programs are essential for this purpose.

To fulfil the above experiments, several fast and efficient algorithms were developed to solve linear ordinary differential equations. These algorithms are programmed in for use in computer algebra system MAPLE® so that CAS users can make use of a large number of terms in the series solution. This enables the users to obtain highly accurate solutions to the ODEs. It was demonstrated that the presently developed algorithms/routines show tremendous improvement over the textbook approach in terms of efficiency and speed.

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