Mathematics and the real world

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'... the enormous success of mathematics in the natural sciences is something bordering on the mysterious and ... there is no natural explanation for it.'

—Eugene Wigner

ONE of the most fascinating features of the intellectual development of the human species is the role that mathematics has played in it, not only as an academic discipline but also as a powerful instrument for understanding the external world and coping with it. In today's popular imagination, this role is captured by images of some absent minded Einstein-like professor whose indecipherable scribblings on the blackboard lead, mysteriously, to astounding practical consequences. But the role of mathematics in human thought is much more pervasive than that. It is not limited just to instances of sophisticated mathematics being used to discuss modern physics, design computers, plan satellite orbits or study the structure of economic models. Down-to-earth race course bookies, far removed from the world of science, rely on elementary probability theory in offering their bets, as do hard headed insurance companies in making their billion dollar profits. In fact practically everyone finds the need to use some mathematics, at least at the level of elementary arithmetic, such as addition, subtraction and multiplication, which are essential for day-to day life. Conversely, illiterate labourers in our country are continually being exploited by employers and shopkeepers because of their inability to perform these simplest of mathematical manipulations.

Thus the need for mathematics is really widespread, all the way from arithmetic in day-to-day life to the most esoteric concepts of topology and group theory employed in physics. But what remains intriguing even after more than two millennia of mathematical applications at different levels is the underlying issue of why mathematics 'works' in the real world and the way in which it works. Indeed, it is not a priori obvious that mathematics should work at all in the physical world, for reasons we will explain shortly. Not only is there no comprehensive or universally accepted theory explaining the success of mathematics in science, but any attempt to unravel this question in some logical fashion is fraught with difficulty. One can get tangled up in issues as varied as the evolution and functioning of the human brain, the role of the environment in developing it, the distinction between the 'inner' and the 'outer' world and so on. Many people have worried about this question, including some great minds, but there has been no unique and established answer. An example of such an attempt is the essay (which contains the quote given at the beginning of this article) entitled 'The unreasonable effectiveness of mathematics in the Natural Sciences' by Eugene Wigner, one of the most distinguished theoretical physicists of the 20th century[1].

Clearly, we cannot attempt here to give definitive answers to this riddle that has been tackled by so many people without reaching conclusive answers. Rather we will share some of our own thoughts on this question. We will also try to give those who have not had the opportunity to ponder on this issue some flavour of why the applicability of mathematics to the real world is indeed puzzling and merits explanation.

Is mathematics a science?

Perhaps one should begin with that old adage that 'Mathematics is the queen of all sciences'. Queen she certainly is. Mathematics represents one of most profound constructs of the human mind and fully deserves royal status. The author views mathematics with respect bordering on awe. But is she the queen of sciences? Is mathematics even a science? Science, whether it be physics, chemistry or biology, is supposed to study what is called the physical or external world 'out there', the entire spectrum varying from celestial bodies to chemicals, metals, gases, living organisms, plants, animals, molecules and sub-atomic particles. The observations and experiments which form the basis of science are expected to have an objective reality of their own, independent of the human observer and his psychological predispositions. (This criterion holds in a fairly straightforward sense for most of science. For systems in which quantum principles play a major role, it still holds in a statistical sense when appropriately generalized to ensembles.)

If science deals with the natural world 'out there', mathematics by contrast seems, at least at the working level, to be a construct solely of human minds. You could put a group of mathematicians in a closed room denied of all contact with the external world except for their pencil and writing pads (many of them do not even need that!) and they can still make progress in their subject and create new mathematics. Correctness or incorrectness of a mathematical statement, given a set of hypotheses, is decided by the fraternity of mathematicians and not by the physical world outside. To quote Keith Devlin, a mathematician who has done a lot of work to popularize the subject, '. . . for all that

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mathematical research feels like discovery, I firmly believe that mathematics does not exist outside of humans. It is something we, as a species, invent. I don't see what else it could be . . .'[2].

How does it happen that a subject like mathematics, seemingly constructed and policed entirely by the 'inner world' of human minds, ends up being such a successful tool in describing and indeed harnessing the external physical world? Is it that the physical world has some intrinsic 'mathematical order', which then instilled in human brains the basic concepts of mathematics and logic through the evolutionary process? In other words, *did the human mind learn about mathematics from the external world rather than the other way around?*

The miracle of mathematical consensus

To pursue this further, consider the closely related issue of how mathematics, even by itself, manages to exist as a human discipline. Let us elaborate on this a little. The very existence of mathematics seems to rely not so much on the external physical world, but on the presence of a common mental understanding among human beings on what is 'logically correct' and what is not, on what 'follows' from a set of hypothesis and what does not. Without such a understanding and universal agreement on logical reasonableness, mathematics cannot exist as a subject. Most times such agreement on the logical validity or otherwise of a proof is immediate among practising mathematicians. Other times, in exceedingly complex mathematical work, the correctness of a step (or series of steps) may not be easy to determine. For instance in the initial 1993 version of Andrew Wiles' proof of Pierre de Fermat's conjecture, there were apparently some gaps in argument discovered by colleagues and reviewers. Of course once this was pointed out Wiles agreed and tentatively withdrew his claim of having proved the historic conjecture. Subsequently Wiles himself was able, with the benefit of related work and suggestions by others, to find ways of proving the missing elements and offered the complete proof. This has since then withstood the scrutiny of the community. But that whole process took about a year![3] (To digress for a moment, this story is one more illustration of the nobility of mathematicians as a community. At least as far as outsiders could tell, it supported Wiles in completing the missing links in his proof, rather than indulge in tasteless quarrels of priority and credit through a whole year of uncertainty about this coveted proof.) From the point of view of our article, this story tells us something about the nature of mathematics – that even in a mental construct as complex as Wiles' proof, a large number of human beings (in fact all mathematicians with sufficient expertise to scrutinize the proof) agreed with one another, first that the original proof carried some extremely delicate flaw and subsequently that the new version was complete.

The interesting question here, and indeed with all mathematical literature and applications is: what is responsible for such agreement between all these people of diverse nationalities and background on such complex issues? If I may be permitted some caricature, what made them all so united in shaking their heads in criticism when Wiles' first proof was scrutinized, and then nodding their heads in agreement when the revised proof came? The same set of people would never show such unanimity in other matters, such as their political views, or their choice of what is the best piece of art or music.

The key players in the Wiles–Fermat theorem came from such diverse origins as England, Europe, the US and Japan. Their early lives and backgrounds were undoubtedly very different. They were not of the same detailed racial material, except in the larger sense of being humans. It is not as if they had some specific life-experience in common, some incident that happened to all of them, which predisposed them all to such agreement. In any case mathematics at the level of Wiles' proof is so abstract and far removed from day-to-day experiences that lifestyles can hardly matter. From where, then, comes such unanimity about which arguments are logically valid and which are not?

In this context it is useful to compare mathematics with music, one of the other great creations of the human mind. As with mathematics, appreciation of music again involves a commonality of taste and sensibility on the part of large numbers of people. But in some sense it is less universal. Music at the highest and most sophisticated level, whether it be Indian or Western classical music, was developed as an abstraction of various folk tunes of the region, religious songs, hymns and other sounds of the local environment. Therefore its appeal can be more localized. In my own experience, I have known many serious scholars and aficionados of Indian music to be quite immune to the greatness of Bach or Beethoven. Conversely I have also known both performers and music professors in the West (particularly prior to the 'sixties before likes of Ravi Shankar and the Beatles led to some 'globalization' of Indian music) to be unable to appreciate Indian music, in part because of its sliding notes, unfamiliar scales and its non-insistence on absolute pitch. Also, as acts of creativity musical compositions are ultimately personal. To quote Devlin again, 'If Beethoven had not lived, we would never have heard the piece we call his Ninth Symphony. If Shakespeare had not lived, we would never have seen Hamlet. But if, say, Newton had not lived, the world would have gotten calculus sooner or later, and it would have been exactly the same!'[4]

Thus even compared to music and other forms of art, all of which are profound human mental constructs and require a commonality of sensibility among vast numbers of people, there is a greater universality about mathematical logic. It is almost as if it has some 'external objective truth' which may take great human cleverness to uncover but whose validity is independent of that individual. The approach towards the proof of a result and its style and notations may be characteristic of that person, but not its core content.

These questions about the very existence of mathematics give us a possible clue on the relation between mathematics and the sciences. This very fact that the same basics of mathematical logic inhabit human minds of all nationalities and cultures may indicate that it is some sort of an inheritance of the collective human experience already at a very early and primitive level. If that is true, it may also make plausible that mathematical ideas, abstracted by human beings from nature in the first place, are then able to help in understanding nature.

Incorrigibility of mathematical results

A completely different family of explanations for why mathematics is so relevant to the physical world could go roughly as follows. In these scenarios, the external world is not necessarily orderly or mathematical but the human mind chooses to study, as quantitative science, only those aspects of the external world that are amenable to man-made mathematical laws, ignoring the rest as 'non-science'. Or alternately one may argue that mathematics is used as a way of classifying physical phenomena and extracting idealizations of them amenable to quantitative analysis. In pursuing this class of explanations, it may be useful to recall Douglas Gasking's essay 'Mathematics and the world'[5,6]. In discussing the fundamental differences between mathematics on the one hand and empirical sciences on the other, Gasking argues that while mathematical propositions are 'incorrigible' by our experience, scientific propositions are corrigible. That is, while scientific laws are constantly open to correction in the light of newer observations, no empirical observation of the physical world can alter any statement considered a truism by mathematicians. The correctness or otherwise of the latter is decided entirely by the 'internal' rules of mathematical logic. Gasking's assertion may sound too strong, and we may need to think long and deep to satisfy ourselves whether it is always true. But simple examples in mathematics already illustrate his assertion.

Consider, for instance, the theorem from Euclidean geometry that the sum of the angles of a triangle add up to 180°. Suppose you want to verify this theorem empirically by drawing a triangle on some surface. The theorem will never be vindicated to 100% accuracy by your measurement. For one thing, all real life measurements will unavoidably be subject to observational errors. Besides, even if you take pains to keep measurement errors very small you may still find that in some situations the sum of the angles of your triangle do not *even approximately* add up to 180°. When this happens a mathematician will not discard or correct Euclid's theorem. Rather, he will point out that the theorem holds only for triangles drawn on planes and that it is your fault that the surface you have used is not planar. (For instance, you might have drawn your triangle on the surface of a football.) The same holds for the familiar Pythagorus theorem which states that the sides of a right angled triangle obey the relation $a^2 + b^2 = c^2$. It too has never been verified with 100% accuracy for the same reasons, viz. that measurements will always have instrumental errors and that the surface on which the triangle is drawn will, in real life, never be a perfect plane. Such empirical non-verifiability of a mathematical theorem to perfect accuracy never leads us to doubt the correctness of the theorem. The theorem is considered correct provided some conditions (hypotheses) are met, conditions which are mentally idealized, but not strictly available anywhere in nature!

This is true even for absurdly simple examples from basic arithmetic involving only integers, so that fractional errors in observation do not come into play. We have the mathematical statement that 20 + 10 = 30. Now, suppose a boy's mother places a plate of 20 hot freshly fried samosas on the dining table for some guests who are expected and a few minutes later brings from the kitchen another batch of 10 more samosas. As per the mathematical result mentioned, one expects altogether 30 samosas on the table. But what if the mother, upon seeing the boy standing nearby with a guilty look, decides to double-check and finds only 28 samosas? Any attempt on the boy's part to wriggle out of the situation by modifying the mathematical proposition to read 20 + 10 = 28 will not work. His mother will consider the original mathematical result 20 + 10 = 30 to be sacrosanct and there may follow a painful investigation, right in front of the arriving guests, into where the 'missing' samosas went.

The above example may appear far too trivial in the context of our serious discussion. After all everybody would agree that there were indeed 30 samosas in existence altogether, as required by the addition rule, and the missing two can be 'accounted for'. That is because we are so accustomed, in day-to-day life, to the total number of objects being conserved. But this need not always be the case. If we were dealing with, say, π^0 mesons instead of samosas, their numbers can fail to add up even without any guilty parties gobbling them up, since the number of π^0 mesons is not conserved (these elementary particles can be created or destroyed through their mutual interaction). If you placed 20 high energy π^0 mesons in an empty box, added 10 more and looked at the system a little later, you may well find 28 or 32 π^0 mesons in the box. But as a mathematical result, 20 + 10 always equals 30. It is in this sense that mathematical propositions are incorrigible no matter what you observe in the physical world. Any deviation from the prediction of a mathematical statement will be attributed to the deficiencies of the physical system (instrumental errors, curvature of the surface, the boy's inability to resist gobbling up samosas, etc.). We have deliberately chosen very elementary examples from basic arithmetic and geometry to illustrate this point in the simplest possible context, but the same holds also for possible physical realizations of any mathematical theorem, however complex.

If mathematical results have a sanctity of their own regardless of whether or not they are accurately realized in physical applications, then how does mathematics end up being so valuable in the real world? The above examples point to one important way in which mathematical results tell us something about a physical system. The failure of an empirical observation to agree with some mathematical prediction can indicate the extent to which the physical system did not accord with the hypotheses that went into the mathematical result. This can be made quite quantitative. For instance, if you draw a very small triangle the difference between 180° and the measured value of the sum of its angles can be used, after correcting for measurement errors, to get a quantitative measure of the local curvature of the surface on which it is drawn.

This also tells us something about the nature of mathematical predictions for the real world. If you had to predict the sum of the angles of a triangle (let us denote it by S) drawn on some unknown surface somewhere in the universe, mathematics would have no prediction at all for it without further information. Those angles may add up to 180° or they may not! It all would depend, among other things, on the curvature of that surface. Thus, a mathematical prediction about any system in the physical world requires other input information about the system. In the above example you need to know the curvature of the surface at every point on it. Now, one way to specify the curvature is to give the measured value of S for all different triangles drawn on the surface! If you do that, then that particular theorem has no predictive content but is only a definition, i.e. a classification index for the curvature of surfaces. In particular, Euclid's result the $S = 180°$ has no predictive value at all unless you know that the surface is a plane. But once a surface is established as a plane to some high accuracy, then all the other theorems of Euclidean geometry do contain many other predictions about triangles, circles and other figures drawn on that surface.

From this one sees that the nature of mathematical applications to any physical system involves correlating different properties of that system with one another through logical connections. Is this giving us some new information about those systems or is it just a case of some hidden tautology? For instance it will really take an infinite number of measurements to ascertain that any finite region of a surface is truly a plane. You would have to check the curvature at every infinitesimal (tiny) sub-region in that region, say, by drawing tiny triangles and measuring the value of S around each point. Therefore to be sure that the numerous theorems of plane geometry are applicable to a given surface you really have to first provide the infinitely many pieces of information that go into ascertaining that it is indeed a plane. Similarly, if you wish to make completely certain predictions with 100% accuracy about the properties of photons and electrons using quantum electrodynamics (QED), you first need to know that the system of electrons and photons does obey the Lagrangian of QED at every space time point, which again amounts to giving as input an infinite amount of information!

In actual practice of course that is not the way we use mathematics for real systems. We use a combination of approximations and 'modelling'. We *assume* that a given physical surface *is* a plane, apply some plane geometric theorems and see if the answers agree with measurements to some acceptable accuracy. Similarly we assume that electrons and photons obey the QED Lagrangian and other postulates of quantum field theory, calculate scattering cross sections using mathematical techniques, and then look for agreement with observation. Clearly this requires the ingenuity of the scientist in finding a good model for a given system, being conversant with the mathematical techniques needed for deriving the consequences of that model and having a large number of samples and observations to verify various predictions.

This also raises the reverse possibility that even discoveries in pure mathematics, at least in its early days, were made possible by the availability of physical systems that were fairly accurate manifestations of the postulates underlying that mathematics. For instance, the theorems of Plane Geometry were discovered by Euclid because the lines, triangles and circles that people dealt with in his time were drawn mostly on planar (flat) surfaces such as table tops. Important early applications of geometry were in land surveys and architecture which again involved plane vertical walls, pillars and land (the earth is flat to a high accuracy within the size-scale of farms and towns). Thus the various theorems of plane geometry were vindicated by numerous empirical observations (within the accuracy of measurements). Once the conclusions of a theorem are empirically obeyed in many cases, then it becomes useful even in those cases where there is some disagreement, as discussed earlier. A piece of land on which Euclidean theorems are not obeyed is presumably not planar, and one could then try to rectify this by land filling and so on.

(But if Euclid had been a creature living on highly curved membranes, soap bubbles and rumpled bed sheets, it is doubtful if he would have discovered his geometrical rules. Of course the theorem that the sum of the angles of a planar triangle add up to 180° would still have been a truism. But, for creatures living on highly curved surfaces, this truism may only eventually emerge as a special case of some more complicated theorems they discover. Similarly if the universe had consisted solely of (highly intelligent) π^0 mesons or photons, whose numbers are not conserved, it is doubtful if they would have discovered our elementary rules of addition and subtraction. For that matter, even the concept of numbers may not have been discovered in such a world where the number of objects has no sanctity!)

Such considerations tend to support the possibility raised earlier, that the content of mathematics (or at least its early simple branches) was developed in response to the physical environment of human beings. Indeed, as we trace the growth of modern physics we can see the parallel symbiotic growth in mathematical methods and results.

The physics–mathematics symbiosis

From its beginning 'modern' physics (i.e. of the seventeenth century and beyond) has been very closely related to mathematics. In fact the organizational separation that prevails between the two subjects in today's academic community was simply not there in the early days of science. Both subjects were viewed as part of the larger pursuit of natural philosophy. Often the same people worked on both areas even at the highest levels. Newton, in order to make precise his laws of motion and obtain exact consequences from them had perforce to also to learn and invent aspects of differential calculus. Similarly in order to calculate the gravitational force of large bodies, starting from the force between point-masses required the development of integral calculus. (Incidentally the controversy between Newton and Leibnitz on credit for discoveries in calculus is well known. But as Herbert W. Turnbull, the British algebrist, points out, long before Newton, another member of the physics pantheon, Kepler, in effect must have used some techniques of infinitesimals in calculating areas bounded by curved orbits. Recall his Law on equal areas swept by planets. He apparently developed these methods in part to estimate the volume of wine-casks![7]) Returning to Newton, although he is generally listed among the great physicists, he was of course also a great mathematician of his time, and the title of his magnum opus was in fact *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy). Similarly, as an undergraduate I first encountered the name of Gauss in connection with physics – the Gauss theorem as used in electrostatics. It is only later in life that I found mathematicians claiming this great man as their own, and as it turns out, with justice, given the huge contributions he had made to pure mathematics. In the mind of the common man too, the distinction between mathematicians and theoretical physicists is blurred. Was Einstein a physicist or a mathematician? To me he was undoubtedly a physicist, but in popular parlance his name is synonymous with mathematical genius!

The further advancement of physics beyond classical mechanics to cover electricity, magnetism, sound and light waves were all made possible by corresponding developments in the theory of ordinary and partial differential equations. The derivations of the results were greatly simplified by the use of complex variable theory. For a long time after that there were continuing instances of physical developments motivating new areas and results in mathematics, such as symmetries in physics spurring the growth of group theory, Brownian motion leading to functional integrals and the Weiner measure (which in turn facilitated Feynman's formulation of quantum mechanics) or most recently the use of non-abelian gauge theory combined with supersymmetry in particle physics set the stage for important results in modern mathematics through the work of Witten and others.

Such a symbiotic mutually supportive growth in our understanding of the physical world on the one hand and our discoveries in mathematics on the other, renders less mysterious the successful application of the latter in understanding and coping with the world. But as mathematics has grown more and more abstract in the past 50 years and farther removed from human experience, instances of such synergy between mathematics and physics become rarer. Correspondingly most of modern mathematics was developed, not from physical examples, but from exerting the power of logical thinking inherent in the mathematician's brain. As we have remarked earlier the basic framework of logic on which all mathematical results rest must be a very general attribute assimilated by the human brain fairly early in human history – or else it could not enjoy the kind of universality it does. It seems incredible that the machinery for constructing the excruciatingly abstract areas of today's modern mathematics could have evolved from primitive human experience of the external world. Besides, even the broad contours of when and how, in biological terms, the 'software' of mathematics (or is it actually the hardware in terms of neural connections?) was loaded into human brains in the evolutionary process are not available. At which stages of human history did the rules of logic and axiomatic deduction, common to all branches of mathematical thinking, get imbedded in human minds? Alternately, if the development of mathematics in humans had nothing to do with the environment, why then is it so successful in describing aspects of the external world?

So we are back to questions with which we had started. We have done no more than elaborate on the issues without arriving at firm conclusions. But, as we said in the beginning, established and universally accepted answers to these questions are not available. We still do not fully understand, to use Wigner's phrase, the 'unreasonable effectiveness' of mathematics.

Mathematics and good taste

We would like to conclude with a somewhat judgmental comment on the use of mathematics in other branches of knowledge. A crucial ingredient in such use in any science or social science is a sense of taste and proportion in the choice of the mathematics to be used. Such taste has not always been displayed and its importance has not been emphasized as much as it should have been. Historically, mathematical methods were first used to any substantial extent only in physics and astronomy. By the end of the nineteenth century, the theoretical underpinning of all of physics and physical engineering was mathematics based. The precision of analysis and accuracy of prediction that this brought to physics were impressive. Gradually but inevitably, this led to a trend among other fields of knowledge as well, of emulating physics in the practice of using mathematical formulations. Today chemical, biological and environmental sciences include many significant sub-fields that employ mathematical techniques. In social sciences, quantitative empirical data and their statistical analysis are used to substantiate and augment qualitative and intuitive theories. In economics, advanced ideas of game theory and even topology come into play. Management theorists employ complicated optimization techniques.

By and large this trend had desirable consequences. More and more subjects were driven by this development towards including quantitative forms of analysis as part of the field. The use of mathematical formulations and equations also induced further precision of thought. But, for all the advantages it offers, there is also a negative side to such widespread use of mathematical methods and 'models' in more and more fields. It can give rise to a misguided impression that a piece of work heavy with mathematical equations necessarily contains results commensurately useful or relevant to the system it has set out to study. Conversely, it fosters a feeling that people who do not explicitly use mathematical methods and symbols are less precise or rigorous in their thinking. Not infrequently this leads to some snobbery associated with the use of mathematical methods. These methods with their mysterious symbols and equations are used to overwhelm others not fluent in them. Such trends can be injurious to the healthy development of a field. They can distort priorities, deflect attention away from the really important issues relevant to the subject matter and should be curbed. It must be remembered that most pioneers and deep thinkers in any field are blessed with intrinsic powers of precision and analysis regardless of the formal level of mathematics they employ. Neither Freud nor Karl Marx nor Darwin employed any mathematical techniques in their gigantic path-breaking work. (One could even argue that it is your 'average good' scientist for whom the mathematical language is more important. It keeps him on the straight and narrow path of logic and prevents him from wandering into vague or internally contradictory statements. Mathematical equations expose such follies.)

As a corollary to this, even in those areas where some mathematics is truly needed and useful, a sense of good taste has to prevail to avoid excess. There is a level of mathematical formulation appropriate to any given problem. Using a more sophisticated version would not only be a case of cracking peanuts with a sledgehammer, but can often obfuscate the real issues. Of course there are many topics, particularly in physics, which genuinely and unavoidably require advanced mathematical machinery. When Dirac first formally codified quantum theory he had to employ the canvas of infinite dimensional vector spaces and operators. There is no significantly simpler, less complicated, mathematics that can comprehensively describe the broad array of new physical and philosophical ideas that quantum theory contains. (The alternate but equivalent formulation by Feynman using path integrals – a method whose core idea was suggested by Dirac himself – is in its own way equally complicated, with basic quantum results like the Heisenberg uncertainty principle requiring very gingerly treatment of the jagged 'paths'). Einstein's General Theory of Relativity again had to unavoidably use the differential geometry of curved 4-dimensional space-time. Similarly, the theory of elementary particles and their strong and electro-weak interactions has to unavoidably use quantum non-abelian gauge fields. All this is fine, as long as the criteria of minimality and unavoidability characterize the choice of the mathematical apparatus used to study a problem. Anything less will not do the job and anything more will be wasteful, if not detrimental.

1. *Symmetries and Reflections – Scientific Essays of Eugene P. Wigner*, Indiana University Press, Bloomington, USA, 1967.
2. Keith Devlin, MAA ONLINE July/August 2001, The Mathematical Association of America, Copyright © 2004.
3. For a short and readable history of the Fermat conjecture see the article by O'Connor, J. J. and Robertson, E. F., available at the website http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Fermat's_last_theorem.html.
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