NON-LINEARITY IN QUANTUM MECHANICS*

BY

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IT is perhaps the most fundamental principle of quantum mechanics that the system of states forms a linear manifold in which a unitary scalar product is defined, and the linear character of the wave function is the superposition principle which is at the basis of quantum mechanics. Developments during the last few years have however given an indication that one may have to admit the possibility of a future non-linear character of quantum mechanics. The first indications of non-linearity came from the positron theory, and later the avowedly non-linear theory of Born was devised to meet the difficulties of infinite self energy in quantum electrodynamics. Still more recently the question of interaction of mesons with nuclear particles appears to suggest that this interaction may be of a non-linear type.

It is instructive to notice that the two fundamental difficulties of quantum mechanics, *viz.*, the question of divergences, and the problem of the elementary particles can be traced in essence to this question of linearity *vs.* non-linearity, the main difficulty in the latter case being the conception of unification of the different wave equations of the elementary particles, light and heavy, into one comprehensive theory if it is to be linear.

Let us look a little more closely at the type of non-linearity that arises in positron theory, and its connection with the non-linear field theory of Born. The simplest application of the former theory is to the adiabatic insertion of an external field into a "vacuum" which induces a sort of vacuum polarisation, in other words, a charge distribution equivalent to the creation of pairs. The next simple application is the case where the external field is so weak that only first order terms in the field strengths and their derivatives need be retained in the perturbation calculations. In this approximation additional terms appear in Maxwell's equations equivalent to saying that Coulomb's law breaks down for distances less than the Compton wave-length. For higher approximations the corresponding field equations are no longer linear, and the superposition principle no longer valid. This deviation from linearity also gives rise to the phenomenon of scattering of light by light which is explained on the basis of the creation of pairs in intermediate states. These features of the positron theory appear to have a close parallelism with the non-linear field theory of Born. In fact, the concept of vacuum polarisation is inherent in the nature of the latter. Further all calculations made on the assumption of the non-linearity in positron theory show that the field equations can be derived from a Lagrangian which, remarkably enough, has the same form as in Born's unitary theory. A quantitative comparison of the two theories shows that the critical field strength of the positron theory is numerically equal to the field strength on the classical Lorentz theory at the boundary of the electron, and this is also the stage at which the feature of non-linearity set in. It is also necessary to observe a striking difference between the two theories. In the theory of the positron negative and positive masses play symmetrical roles while Born's theory does not envisage negative mass even abstractly. Now, Pauli has recently shown on very general considerations that there is an intimate relationship between spin, and definiteness of charge or energy density. Thus the definition of a definite particle density (4-current) which transforms like the components of a 4-vector is not possible for integral spin, and a positive definite energy density, and also a positive definite total energy are not defined for half-integral spin. These, however, are not positive statements since they do not say that for integral spins there is a definite energy density, and for half-integral spins a definite charge density. In fact, this is no longer the case for spins greater than 1. It is only in the case of the small spins 0, ½ and 1 that such positive statements could be made. Thus the spin ½ is characterised by the fact of the possibility of a definite charge density, and spins 0, 1 of definite energy

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density, and thus these spins are distinguished from all other higher spins. As is well known, Born's theory does not explain the spin of the electron correctly giving integral eigen values for the spin operator. This fact which is in consonance with the energy density being positive appears as a consequence of the fundamental assumption of non-linearity introduced in the theory. On the other hand, the positron theory which explains spin correctly has also non-linear features which appear to have been forced into the theory. Thus, in spite of the similarities between the two theories, it looks as if non-linearity as a fundamental physical principle does not perhaps really go very deep being more in the nature of a mathematical device.

As another example of a theory wherein non-linearity appears as a secondary feature might be mentioned the recent classical theory of the electron given by Dirac. This theory retains Maxwell's equations as they are, *i.e.*, the field equations are linear, and the infinities are avoided by a sort of subtraction formalism of the type used in the positron theory, getting the finite mass of the electron as the difference between the infinite negative mass at the centre, and the infinite positive mass of the surrounding Coulomb field. The equations of motion of the electron in the electromagnetic field, however, come out non-linear in the theory, giving an example of non-linearity of a derived or secondary type.

We might now consider the non-linearity that arises in the theory of the meson in connection with the interaction of mesons with nuclear particles. The meson theory originally proposed by Yukawa for the explanation of the exchange character of nuclear forces is based on the heuristic assumption that nuclear forces can be described by some sort of linear field. The theory has been further refined by the introduction of the vector formalism, but the field is still linear differing, however, from the Maxwell field in two important respects, *viz.*, the meson field itself is charged, and is characterised by a "meson charge" g_1 and a "meson dipole moment" g_2. Thus taking into consideration the mass of the meson we have three essential differences between the theory of the electron and the meson. When the meson theory is applied to the interaction with nuclear particles, as for example, to collision processes with large cross-sections, it appears to break down for distances less than $1/\lambda$ ($= h/\mu c$; $\mu =$ mass of meson). There appear terms in the interaction which increase with increasing energy. Connected with the same behaviour of the interaction is the fact that many other effects as, for example, the perturbation of the self energy of the proton diverge more acutely than in ordinary radiation theory. This has led some theoretical Physicists to doubt the correctness of the fundamental equations even for mesons of energy comparable with their rest mass, and to take the position that these facts set a limit to the applicability of quantum mechanics suggesting that the right interaction may be of a non-linear type. However the recent work of Bhabha on the classical theory of neutral mesons, and some as yet unpublished work by the same author on the classical theory of point dipoles seem to suggest that these difficulties are really spurious and arise because of the fact that radiation reaction is not properly taken account of in quantum mechanics. In any case, even if such a non-linear feature becomes inevitable, we should observe that the position regarding non-linearity here is different from that in quantum electrodynamics where the deviation from linear laws are purely of a theoretical nature, and cannot be verified experimentally. Thus all radiation effects of an electron could very well be treated by a first order approximation theory, and there would be no experiments contradicting such a theory. Here the behaviour of mesons in traversing matter provides direct experimental tests for the interaction in question, *i.e.*, the non-linearity is of a determinable type.

Mention might also be made here of another attempt at a non-linear theory based on the idea of a new universal constant like an absolute length or momentum or field strength. Basing his considerations on the fact that one has to rely only on theoretic grounds for introducing non-linearity, Born has recently introduced his principle of reciprocity. According to this, the only natural and unique way of introducing non-linearity into a field theory is that used by Einstein in his theory of gravitation, namely, the postulate of general invariance which leads one to consider space-time as "curved". But Einstein's theory has to do with very small curvatures imperceptible in the region of laboratory dimensions. It is also clear from the smallness of the gravitational constant that cosmological curvature has nothing to do with atomic effects. These latter are bound to be extremely small, of the order r_0. But small r_0 means large momentum $b = h/r_0$. Born's principle is that the domain of elementary particles has to be considered from the standpoint of momentum-space in which a non-linear geometry with small curvature reigns. The curvature equation should contain solely r_0 and b (or r_0 and h) but not the gravitational constant.

The question whether the linear or non-linear character of a field theory is related to the question of derivation of the equations of motion from the field equations alone might now be examined. For instance, in the well-known Helmholtz theory of vortices in a non-viscous fluid the motion of line-singularities is actually determined by the partial differential equations alone which are there non-linear. In the gravitational theory where also the field equations are non-linear the recent "approximation method" of Einstein-Infeld-Hoffmann shows that such a thing is also possible. On the other hand, in the non-linear theory of Born the equations of motion are not determined by the field equations alone.

Coming to linear field theories we might notice that in ordinary Maxwell's equations in which electrical particles are regarded as point singularities, the motion of these singularities is not determined by the linear field equations. On the other hand, in the recent Dirac theory of the electron the equations of motion, though themselves non-linear, follow immediately from the field equations which are linear.

We have described a number of non-linear theories in some of which non-linearity enters as a fundamental factor, and in some others as a secondary feature. Again in some it is purely of a theoretic origin and in others of a type which can be decided by observation. As regards non-linearity as a principle in theoretical physics it is difficult to say anything in general. Judged by the principle of simplicity linear laws appear to be the most suitable ones, and moreover they have the advantage of uniqueness.

SOIL EROSION CAUSED BY WIND

BY

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SOMEWHERE above the dust, gloom and grit of the dust-storm, the sun is shining. For want of adequate protection on some land surface, the wind has whipped up the loose soil and carried it many miles to make life hideous for the dwellers in some distant city. Somewhere it will come to rest. Somewhere wind-transplanted seeds may lodge and grow, but the sower will not be the reaper.

We have been asked to explain the significance for India of wind erosion. Roughly one may say that wherever there are dust-storms there is wind erosion; a fairly large part of India is suspect, but without a detailed survey of soil behaviour actually under the stress of exceptionally high winds, it is very difficult to give a concise picture of the damage which may be occurring in different parts of India.

In soil erosion caused by water, it is the exceptionally severe storm spread over many hours, but with interspersed intervals of torrential downpour, which causes infinitely more damage than weeks or months of ordinary rain showers. So also in wind erosion, it is the sustained and exceptional storm which does the greatest damage, but here the phenomenon is of very high wind following weeks of drought, and possibly assisted by widespread ploughing operations or forest fires or excessive grazing, all of which leave powdery soil exposed to "dust devils" which carry it into the upper air.

Erosion by water and by wind is frequently found acting in turn on the same patch of ground, but roughly speaking, water erosion is a serious phenomenon on slopes of more than 3 per cent. whereas wind erosion is more prevalent on flat or slightly undulating surfaces. Nature sees to it that where water cannot punish man for his destructiveness and poor husbandry, the wind can do so. A land surface typical of wind erosion is carved out in hummocks of soil, the crest of each rise being built round some deep-rooted vestige of the previous plant cover. Round each of these centres the wind has picked away the exposed soil until quite deep channels have been formed with scallopped markings on the sandy surface reminiscent of wave action ripple markings on a sea beach. Such a landscape is well seen a few miles north of Campbellpur in the Attock District. In many places where a typical wind erosion landscape has not developed, however, wind erosion may nevertheless be serious, as can be judged from the way wind borne sand and powdery grit accumulates behind Euphorbia and other live hedges. Mr. J. A. Wilson of the Forest Service recently quoted as examples the land along the Coimbatore–Bolanpatti and Coimbatore–Mettupalyam roads, and observant readers can doubtless add their own examples from their own country-side.

The loss from wind erosion is an even more insidious one than in water erosion, for the lightest particles in the soil are inevitably the ones you want to keep, the infinitely small grains of manurial material and partially leached salts. The factors which contribute to the severity of wind erosion are bare earth surfaces of pulverised soil, and a high wind with an unobstructed stretch for it to operate in. Eliminate any of these and the wind cannot collect a large load. First, then, keep the plant cover intact even if it is only an incomplete carpet of bunch grasses. Secondly, where ploughing is to be done, keep your fields under diversified crops in strips set at right angles to the prevailing wind. The actual velocity of the wind cannot of course be controlled, but by ridging the ploughed soil in plough land, soil drift can be greatly reduced. Thirdly, make full use of nature's own prescription by growing shelter-belts of trees and tall grasses.

Possibly the closest study of shelter-belts has been made in Russia, where recent planting under the second Five Years' Plan has amounted to 865,000 acres of shelter-belt, a truly gigantic scheme vying in scope with the American Great Plains Shelter-belt project. The efficiency of these shelter-belts across the wide and wind-exposed plains has been thoroughly tested and measured. They cause a marked increase in productivity in each field in a strip up to 20–25 times the height of the trees in the belt, even when the belt consists of a single line of trees. They not only reduce the extreme severity of the wind and stop it from transporting soil, but the temperature and aridity extremes are reduced so that the crop ripens more regularly. In snowy areas the moisture added to the ground by the deep and regular snow beds in rear of each belt is of the greatest value and contrasts markedly with unprotected land where the snow beats along in a blizzard and eventually leaves much of the ground frozen, uncovered and dry. Similar results are also reported from Canada and Hungary.

Soil is most in danger from wind when it is pulverised and broken down. When its natural aggregation into a crumb structure has been preserved and encouraged by good methods of husbandry, the danger is greatly reduced. When a large flat open area has been badly cultivated and starved of manure so that the crumb structure has collapsed into a friable powder, the wind can produce chaotic damage in a very few days or even hours. The pulverised soil is lifted bodily from the surface of the fields and dumped around obstructions such as hedges, buildings, roads and railways where it is least welcome. The vagaries of such damage are often hard to understand, for the top few inches of fertile soil may be removed from one man's fields while another's field near by is buried under a load of sterile sand. Soil thus lifted up is itself a potential destroyer, for it has a sand-papering effect upon any vegetation standing in its path, and in very severe dust-storms the vegetation may be stripped or browned off almost as if frost or hail had destroyed it.