

A quadrature method for computing nonlinear source term due to wave–wave interactions

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An accurate method using composite Gauss–Legendre N -point quadrature formula is presented for solving the nonlinear wave–wave interaction source term in deep waters. This method employs a polar grid in the wavenumber space with a constant geometric ratio λ and uses the scaling relation for the transfer integral. The accuracy of the method can be tested for different λ s by increasing N . This increase in λ will help in calculating the nonlinear source term at less number of frequency points, resulting in reduced computation time. We also included the procedure for obtaining the nonlinear results at more number of frequency points. A study of 1D nonlinear source term $S_{nl}(f)$ with results of exact methods, indicates that the present results are comparable and qualitatively in good agreement with Webb–Resio–Tracy (WRT) method in spite of slight differences at higher frequencies. This slight difference is due to different polar grids employed for the input vectors in the present method, whereas they are the same in the WRT method. The present method thus considers possibilities that are not explored in the WRT method.

Keywords: Nonlinear interactions, resonating quadruplets, quadrature.

NONLINEAR wave–wave interaction source term plays an important role in the area of wave models, ever since its contribution to the wave model has been recognized with pioneering studies by Hasselmann¹, who applied perturbation theory up to fifth order and obtained the energy transfer rate between four-wave components for finite depth water. It is the contribution of the energy transfer between sea waves due to nonlinear interactions that explains the peak enhancement and the shift of the peak to lower or higher frequencies.

Numerical methods for solving the nonlinear source term are being tried by several researchers and scientists involved in the development of the wave model, based on parametric methods or other types of approximations to improve computational speed. Owing to the mathematical complexities and extreme computation time involved, the methods are not convenient in any operational wave model. Webb² has developed an integration approach that

evaluates the nonlinear source term. Another significant contribution to the nonlinear computation is the EXACT-NL method by Hasselmann and Hasselmann³. Significant progress in the operational wave prediction models (like WAM⁴, WAVEWATCH⁵ and SWAN⁶) came with the work of Hasselmann and Hasselmann^{7,8} who proposed a computationally fast and approximate method called Discrete Interaction Approximation (DIA) to solve the nonlinear problem in deep waters. By introducing DIA, Hasselmann and Hasselmann provided the possibility of having a fully discrete spectral wave model that is based on solving the energy balance equation, which was not possible earlier. On the other hand, it should be emphasized that in the DIA, heuristic strategy is used based on the assumption that in practical applications which include wave growth and dissipation from the wind, it is possible to model the contributions to the source function approximately from resonant wave–wave interactions, using a representative form of wave–wave coupling. In particular, in the initial formulation, a minimal number of (quadruplet) resonant interactions were used. Despite this limitation, by introducing some fine-tuning, based on a parameterization of wave dissipation and growth from the wind, using DIA and a wave propagation model (WAM), it became possible to reproduce reasonable deep-water wave growth curves. From approximate scaling relationships^{9,10}, it was also possible to extend the initial formulation and to more accurately evaluate the source function, both exactly and using the DIA. These developments also provided a means for approximately extending the DIA to finite depth conditions.

In the present work, we suggest a procedure for evaluating the nonlinear source function using Gauss quadratures. The significance of the approach is that through this kind of algorithm, it is possible to repetitively integrate particular kernels, in the associated integral equations, at a minimal number of spectral points (associated with the values of the wave-vector), in a way that can be extended (through scaling relationships) globally to many spectral points. In principle, the procedure should make it possible to determine the source function considerably more efficiently than has been possible previously. The associated algorithm could be used both in the context of approximate schemes, such as the DIA, or in the context of more precise representations that are used in the evaluation of the exact nonlinear source function.

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Alternative techniques are being tried continuously for a more general method that is fast and accurate. An accurate method can be achieved with the use of Gauss quadratures, as it is a global one and hence is expected to provide better accuracy in most higher-dimension problems. It also requires only less number of points and is independent of the geometry of the domain.

The motivation for this study is to find an accurate, general method to evaluate the nonlinear wave-wave interactions. It is not an attempt to develop a computationally fast accurate method for solving the nonlinear problem. Here, we present a new formulation to solve the nonlinear problem in deep waters using Gauss–Legendre quadrature method and employing scaling relations for the transfer integral that are inherent in the collision process. This method can however be improved for a computationally faster method and hence more detailed results concerning the growth curves will be considered in our next paper. The results of 1D source term $S_{nl}(f)$ versus frequency f are presented for the deep water case by considering an input spectrum such as Jonswap Product Spectrum and compared with results of exact methods such as Hasselmann’s EXACT-NL method and Webb–Resio–Tracy (WRT) method, as described by Resio and Perrie¹¹. Besides the illustration of the convergence test of the method, results for the 1D nonlinear source term for less number of points in the wavenumber polar grid are also considered.

Computation of nonlinear source term

Following Hasselmann¹, the nonlinear source term in deep waters can be written as

$$\frac{\partial A_1}{\partial t} = \iiint C(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) D(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) \delta(\vec{K}_s) \delta(W) d\vec{K}_4 d\vec{K}_2 d\vec{K}_3. \tag{1}$$

Here $\vec{K}_i = (k_i, \theta_i)$ and $A_i = A(\vec{K}_i)$ are the i th interacting vector wavenumber and action density at \vec{K}_i respectively. $W \equiv \omega_1 + \omega_2 - \omega_3 - \omega_4 = 0$ and $\vec{K}_s \equiv \vec{K}_1 + \vec{K}_2 - \vec{K}_3 - \vec{K}_4 = 0$ represent the wave frequency and wave vector resonance conditions respectively. The terms $C(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4)$ and $D(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) = [A_1 A_3 (A_4 - A_2) + A_2 A_4 (A_3 - A_1)]$ represent coupling coefficient and density function respectively.

Equation (1) may be reduced by integrating over \vec{K}_4 resulting in

$$\frac{\partial A_1}{\partial t} = \int T(\vec{K}_1, \vec{K}_3) d\vec{K}_3, \tag{2}$$

where

$$T(\vec{K}_1, \vec{K}_3) = \frac{1}{\pi} \int_0^{2\pi} d\theta_2 C(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) D(\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4) |\nabla W|^{-1} \tag{3}$$

is the nonlinear transfer integral. The inverse of the gradient term, $|\nabla W|^{-1}$, is to be evaluated at each resonating quadruplet $\{\vec{K}_1, \vec{K}_2, \vec{K}_3, \vec{K}_4\}$. Equation (3) was obtained using:

(i) the following properties of the Dirac delta function

$$\delta[f(\vec{r})] = \frac{\delta(\vec{r}-\vec{r}_0)}{|\nabla f|_{(\vec{r}_0)}}, \quad f(\vec{r}_0) = 0, \tag{4}$$

$$\delta(\vec{r}-\vec{r}_0) = \frac{\delta(r-r_0)}{\pi r} \quad (\text{in polar form}), \tag{5}$$

$$\int \delta(r-r_0) f(r, \theta) dr = f(r_0, \theta), \tag{6}$$

where $\vec{r} = (x, y)$, $\vec{r}_0 = (x_0, y_0)$ and $r = |\vec{r}|$,

and (ii) integrating over the radial direction of \vec{K}_2 . In the present article, the transfer integral is solved using a composite trapezoidal rule. Singular points (which may arise when the magnitudes of the input vectors \vec{K}_1 and \vec{K}_3 are equal) of the transfer integral are naturally avoided when this quadrature method is used. A detailed description on the existing methods for obtaining resonating quadruplets can be found in Van Vledder¹².

From the computational point of view, it was shown advantageous to choose $k_3 > k_1$ as the frequency resonance condition requires less number of points on the locus using the polar method by Prabhakar and Pandurangan¹³. In the computation of the nonlinear source term, the locus (of \vec{K}_2) was obtained in the anticlockwise direction for an input angular increment on it with the starting point on the line of symmetry. Also, when $k_3 < k_1$, the same number of points on the locus (as for $k_3 > k_1$) have been used, which can be obtained from the locus results for $k_3 > k_1$ using a simple transformation.

Let the magnitude of the input wave vectors \vec{K}_i ($i = 1, 3$) lie in the interval $[k_0, k_u]$, where k_0 and k_u are the minimum and maximum wavenumbers respectively. Divide this interval into I sub-intervals with geometric ratio λ . Here the value of k_0 is chosen as 0.14 m^{-1} . In general, $k_u \rightarrow \infty$.

Applying composite Gauss–Legendre quadrature formula to the integral in eq. (2), we finally get

$$\frac{\partial A(k_{1,p}, \theta_1)}{\partial t} = \Delta \theta_3 \sum_{j=1}^J \left[\frac{(\lambda - 1)}{2} \sum_{n=1}^N \left(w_n \sum_{i=1}^I \alpha_{3,i} T(k_{1,p}, \theta_1; k_{3,i,n}, \theta_{3,j}) k_{3,i,n} \right) \right]; \tag{7}$$

$p = 1, 2, \dots, I,$

where $k_{3,i,n}$ are the abscissas of the Legendre polynomial of order N , in the sub-interval i , w_n are the weights associated with the abscissas; $k_1 = k_{1,p} = \lambda^{p-1}k_0$, $\alpha_{3,i} = \lambda^{i-1}k_0$, and $\theta_{3,j} = \theta_1 + (j - 1)\Delta\theta_3$.

An advantage in the present method is that we can write $k_{3,i,n} = \lambda^{i-1}k_{3,i,n}$. Since the present method uses a low order high-accuracy quadrature method, it is expected to give highly accurate results when N is increased.

With the use of scaling relationships in deep waters first shown by Tracy and Resio¹⁰,

$$C(\vec{\mathbf{K}}_1', \vec{\mathbf{K}}_2', \vec{\mathbf{K}}_3', \vec{\mathbf{K}}_4') = \lambda^6 \cdot C(\vec{\mathbf{K}}_1, \vec{\mathbf{K}}_2, \vec{\mathbf{K}}_3, \vec{\mathbf{K}}_4) \tag{8}$$

$$|\nabla W' \Gamma^{-1}| = \sqrt{\lambda} \cdot |\nabla W \Gamma^{-1}|$$

that are inherent in the collision integral for each combination of four wavenumbers satisfying $(\vec{\mathbf{K}}_1', \vec{\mathbf{K}}_2', \vec{\mathbf{K}}_3', \vec{\mathbf{K}}_4') = \lambda(\vec{\mathbf{K}}_1, \vec{\mathbf{K}}_2, \vec{\mathbf{K}}_3, \vec{\mathbf{K}}_4)$, the successive transfer integral becomes

$$T(\vec{\mathbf{K}}_1', \vec{\mathbf{K}}_3') = \lambda^{13/2} \left(\frac{1}{\pi} \int d\theta_2 C(\vec{\mathbf{K}}_1, \vec{\mathbf{K}}_2, \vec{\mathbf{K}}_3, \vec{\mathbf{K}}_4) \right) \left(D(\vec{\mathbf{K}}_1', \vec{\mathbf{K}}_2', \vec{\mathbf{K}}_3', \vec{\mathbf{K}}_4') |\nabla W|^{-1} \right) \tag{9}$$

using eqs (3) and (8). The present method uses the factor $\lambda^{13/2}$ for the computation of successive transfer integral, whereas the factor $\lambda^{15/2}$ was used in the works of Tracy and Resio¹⁰, and Resio and Perrie¹¹. This is because, arc length was used as the integration parameter in their works instead of the argument θ_2 . In shallow waters, however, eq. (9) cannot be used.

Having calculated $\partial A(\vec{\mathbf{K}})/\partial t$, the 1D nonlinear source term $S_{nl}(f)$ can be computed using $S_{nl}(f) = \int S_{nl}(f, \theta) d\theta$, where $S_{nl}(f, \theta) = (4\pi\omega^4/g^2)(\partial A(\vec{\mathbf{K}})/\partial t)$. The units of frequency (f) and nonlinear source term ($S_{nl}(f)$) are Hz and $m^2/Hz/s$ respectively. The method is illustrated in the results and discussion by considering JONSWAP spectrum with a cosine-squared distribution. The spectral parameters such as Phillips parameter $\alpha = 0.01$, peak frequency $f_n = 0.3$ Hz, frequency spread factor

$$\sigma = \begin{cases} 0.07, & f < f_p \\ 0.09, & f \geq f_p \end{cases}$$

have been considered throughout. The input values for the spectral peak enhancement factor γ are shown in the respective figures.

Results and discussion

In all the figures shown below, the angular resolutions $\Delta\theta_3$ and $\Delta\theta_2$ are chosen to be the same. All the input parameters used are clearly mentioned in the respective figures.

Convergence tests for the present method

Figure 1 shows the convergence test of 1D nonlinear source term $S_{nl}(f)$ versus frequency f for the PM spectrum. In this case, the angular resolutions are chosen to be the same. That is, $\Delta\theta_1 = \Delta\theta_3 = \Delta\theta_2 = 9^\circ$. As is obvious, there are no significant changes in the results for increasing N . Since no simplifying assumptions are made, accuracy can be improved by increasing the parameter N . It is found that the computed results are correct to 5 and 6 decimal places, when N is chosen as 2 and 3 respectively. It is found that the computation time of the method is of the order of N . Convergence can also be established for $N = 1$ with finer inputs for $\Delta\theta_3$ and $\Delta\theta_2$.

Figure 2 illustrates the convergence test of 1D nonlinear source term $S_{nl}(f)$ when the angular resolutions of the integration direction parameters are chosen to be the same. That is, $\Delta\theta_3 = \Delta\theta_2 = 3^\circ$. In this case, the angular resolutions of the input wave vectors are chosen to be different (i.e. $\Delta\theta_3 \neq \Delta\theta_1$). It is clear that convergence is established for $N = 1$ even to lower resolution ($\Delta\theta_1 = 30^\circ$) case. The computation time for $N = 1$ is in the order of hours on the P4 system. For the same integration parameters, the computation time doubles when $\Delta\theta_1$ is reduced by half.

Comparison of integration results to previous estimates

Figure 3 a–d compares the nonlinear transfer due to wave–wave interactions obtained from our integration method with results of Exact-NL method and WRT method (with sector grid) taken from Hasselmann and Hasselmann³, and Resio and Perrie¹¹ respectively. The nonlinear transfer results of WRT method with circular grid are also included

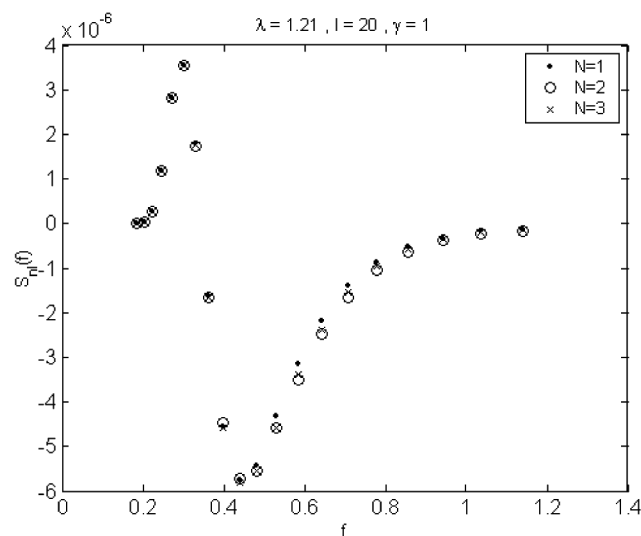


Figure 1. Convergence test for 1D nonlinear source term when $\Delta\theta_1 = \Delta\theta_3 = \Delta\theta_2 = 9^\circ$.

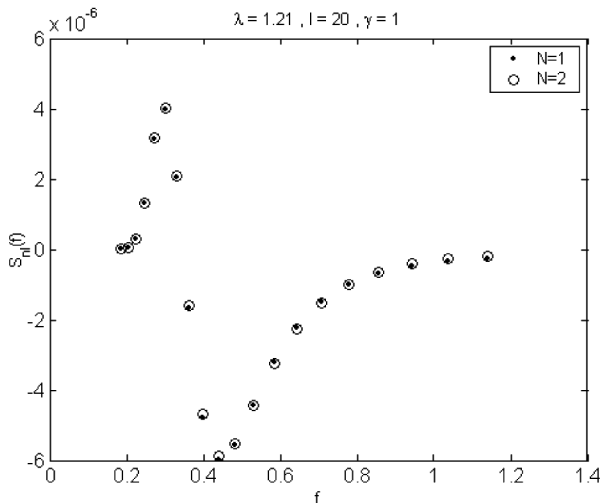


Figure 2. Convergence test for 1D nonlinear term when $\Delta\theta_3 = \Delta\theta_2 = 3^\circ$ and $\Delta\theta_1 = 30^\circ$.

for comparison. In all these figures, we used $\Delta\theta_1 = 15^\circ$ as reported in Hasselmann and Hasselmann³. The angular resolutions of the integration (direction) parameters are as considered in Figure 2. It may be noted that the integration parameters need not be the same for other methods such as WRT method. This is because of the distribution of points on the locus which differs with the polar methods. The angular resolutions are chosen to be high so as to find the sensitivity of the nonlinear results when compared with different methods. The parameter N was chosen as 1 in view of the convergence of the method shown in Figure 2 and finer value for $\Delta\theta_1$ chosen. In Figure 3 *b*, the directional distribution corresponds to case 3 of Hasselmann and Hasselmann³.

Similar to WRT method, all results obtained using the present quadrature method are smooth. A comparison with the results of exact methods: Exact-NL and WRT, indicate that the present results shown in Figure 3 *a-d* are comparable and reasonably closer to WRT method in spite of slight differences at higher frequencies. It may be noted that results of WRT method obtained with sector grid and circular grid agree well, even though slight differences can be seen due to the choice of the polar grid. An important observation is that in Figure 3 *c*, there is no third lobe (compared with existing methods) with the use of the quadrature method. This deviation in the results may be due to higher angular resolution for $\Delta\theta_3$ employed in the present method. It may be noted that the angular resolutions $\Delta\theta_1$ and $\Delta\theta_3$ of the input polar grids are same in the WRT method. More precisely, the number of points on the \mathbf{K}_1 and \mathbf{K}_3 polar grids are same. In the present method, however, they are chosen to be different. It may also be noted that we can increase the resolutions of the integration direction parameters (i.e. $\Delta\theta_3$ and $\Delta\theta_2$) independently. In the present method, $\Delta\theta_3$ is chosen dependent on $\Delta\theta_1$.

Reduction in computation time

Since the computation time is in the order of hours for high resolutions of the integration direction parameters (such as considered in Figure 3), the method is slow compared to existing methods such as WRT method. It is however possible to reduce the computation time (to order of few minutes) with the present quadrature method by considering less number of points (i.e. less frequency and angular points) in the wavenumber polar grid, while at the same time maintaining accuracy. This is illustrated in Figures 4 and 5 for the same input considered in Figure 2, but I is now varied.

Figure 4 shows three curves of the nonlinear source term $S_{nl}(f)$ for $N = 1$. The first curve shows the results at 8 frequency points corresponding to $I = 8$. The second and third curves show the results at 5 and 4 frequency points corresponding to $I = 5$ and $I = 4$ respectively. The values of λ for the first, second and third curves are found to be 1.6105, 2.1436 and 2.5937 respectively. The computation time for the third curve is about 5 minutes. Here, the nonlinear term is calculated at the reduced number of frequency points for the same interval $[k_0, k_u]$. It may be noted that the maximum wavenumber k_u is kept fixed throughout Figures 1–5 for the present work.

Obviously, the first curve contains all the frequency points of the third curve, whereas it contains only the starting frequency point of the second curve. By comparing first and third curves at the common frequency points, it is clear that there are deviations in the nonlinear results.

As N is increased, these deviations in the nonlinear results tend to reduce. This is illustrated in Figure 5, the curves of which correspond to the corresponding curves of Figure 4 but with increased N . A comparison of the first and third curves in Figure 5 clearly indicates that the results of the nonlinear term are in good agreement at the common frequency points. The results of the nonlinear term shown are accurate to 6 decimal places at the common frequency points. Thus by reducing the number of points and increasing N , the accuracy can be improved. But reducing the number of points results in the increase of the value of λ . Hence it is shown that more the value of λ , N needs to be increased for better accuracy.

The second curve of Figure 5 considers the nonlinear results at those frequency points which do not coincide with any of the frequency points of the first curve except at the end points. By joining the successive points of the first and second curves, we get a finer result for the nonlinear source term, as there are no common interior points and the results coincide at the starting point. In this case, two sets of points are considered and the method is applied to each set. We may thus consider as many finite sets of (non-overlapping) points as necessary, which coincide at the end-points and apply the method to each set. A finer result for the nonlinear source term is then obtained by joining successive points.

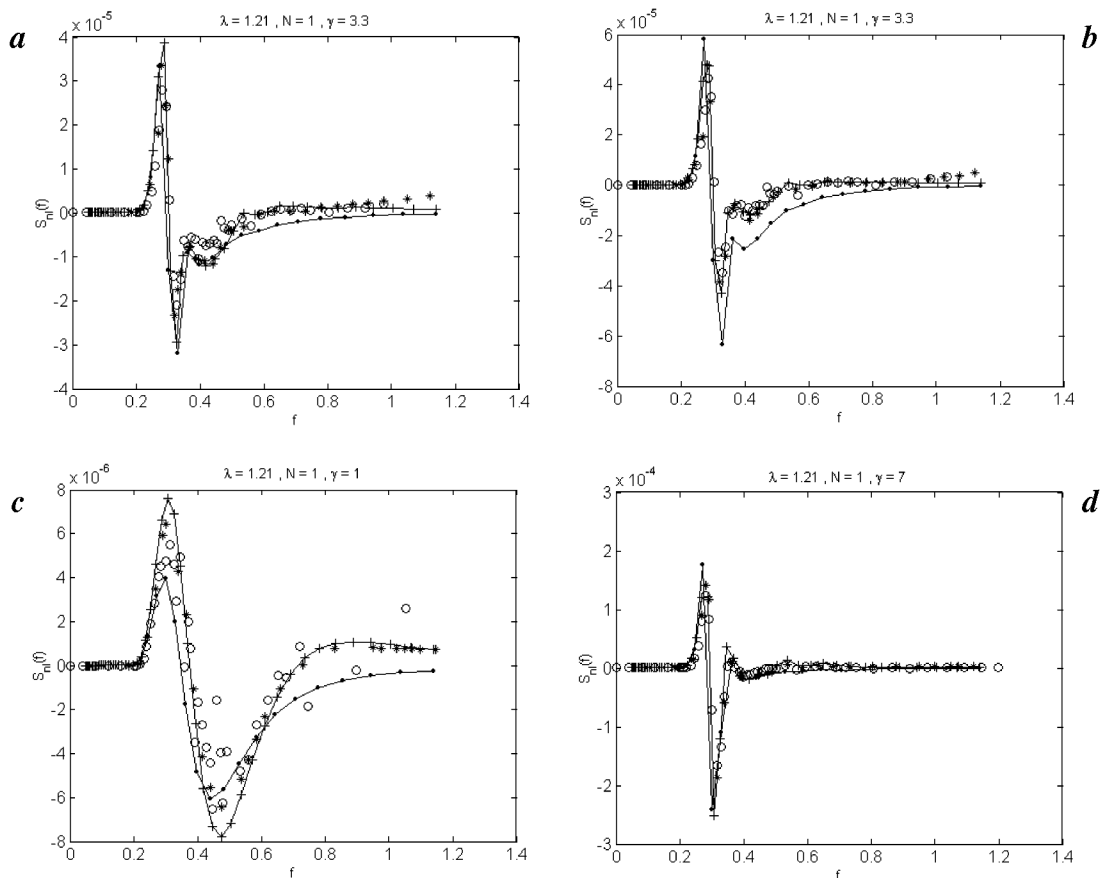


Figure 3. Nonlinear transfer due to wave-wave interactions obtained from the integration method of this article. — Compared to WRT method (sector grid); *, WRT method (circular grid); +, —, and Exact-NL method; o. *a.* For case 2 of Hasselmann and Hasselmann³. *b.* As in (*a*) for case 3 of Hasselmann and Hasselmann³. *c.* As in (*a*) for case 13 of Hasselmann and Hasselmann³. *d.* As in (*a*) for case 15 of Hasselmann and Hasselmann³.

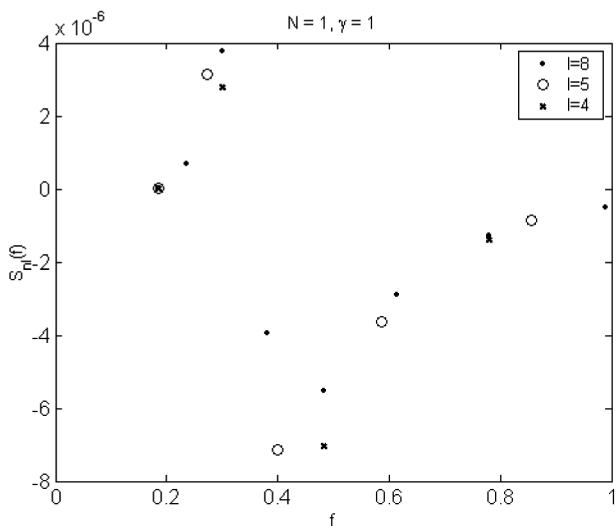


Figure 4. 1D nonlinear term for input as in Figure 2, but with different *l*.

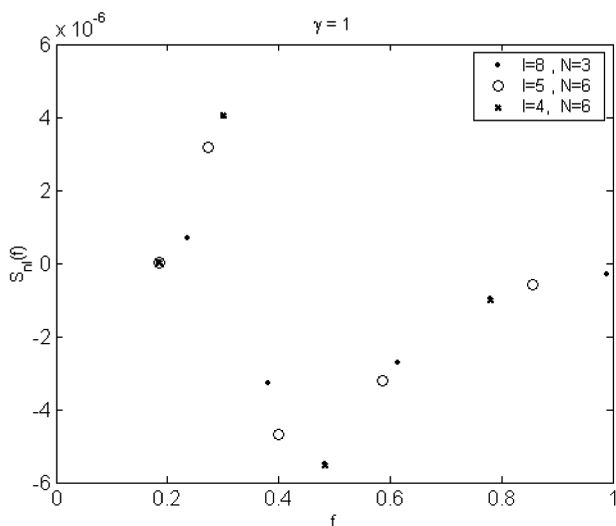


Figure 5. 1D nonlinear term for input as in Figure 4, but with increased *N*.

Figure 5 thus demonstrates the procedure for studying the nonlinear wave-wave interaction problem accurately

with this quadrature method at less (and as well as more) number of points in the wavenumber polar grid for a

given input spectrum. The present method may be useful to researchers in better understanding this nonlinear problem.

Summary

In this article, a quadrature method is employed for solving the nonlinear wave–wave interaction source term in deep waters. The resonating quadruplets (locus results) required in the estimation of transfer integral are computed using the polar method by Prabhakar and Pandurangan¹³. A comparison study (shown in Figure 3) of nonlinear source term $S_{nl}(f)$ with results of exact methods: Exact-NL and WRT, indicate that the present results are comparable and reasonably in good agreement with the WRT method in spite of slight differences at higher frequencies. An important observation is that there is no third lobe (in Figure 3c for the input PM spectrum when compared with existing methods) in the results of the nonlinear source term with the use of quadrature method. This deviation may be due to finer angular resolution for $\Delta\theta_3$ chosen in the present method, which was not explored in the WRT method. More precisely, we chose $\Delta\theta_3$ to be finer than $\Delta\theta_1$, which were considered to be the same in the WRT method. The present method thus considers two different polar grids for the input wave vectors and is particularly helpful in computing the nonlinear results at less number of points in frequency–direction polar grid which may be checked with existing exact methods.

Figure 5 demonstrates the procedure for studying the nonlinear problem accurately with this quadrature method at less number of points and at more number of points respectively, in the polar grid for a given input spectrum.

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