

# Science frontiers – semigroups of maps on operator algebra, their dilations and applications

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**The structure of a semigroup of normal completely positive maps on a  $*$ -algebra and the associated (stochastic) dilation is studied, first in the case when the semigroup is uniformly continuous and secondly when it is only strongly continuous. In the second case, some further structure and assumptions are necessary. An application in the physical problem of a damped harmonic oscillator is also described.**

**Keywords:** Fock space, operator algebra, science frontiers.

## Introduction

The study of the algebra of operators in a Hilbert space in a systematic manner began with the pioneering work of von Neumann and Murray<sup>1</sup> and has scaled great heights with the works of Tomita and Takesaki<sup>2</sup> and of Connes<sup>3</sup>. Many of the mathematical questions in these path-breaking investigations had their origin in physics, more precisely quantum mechanics. For example, the problem of the classification of the von Neumann factors involves study of dynamical systems in an operator algebra while the theory of Tomita and Takesaki has a natural canonical relation with the so called ‘Kubo–Martin–Schwinger’ condition of the equilibrium quantum statistical mechanics.

One of the early questions in quantum mechanics that was answered completely by Wigner in the 1930s is the following.

*Wigner’s Theorem:* Let  $\mathcal{H}$  be a complex Hilbert space of dimension  $\geq 3$ . Then to every automorphism of  $\mathcal{B}(\mathcal{H})$ , the von Neumann algebra of all bounded linear operators in  $\mathcal{H}$ , there corresponds (uniquely up to multiplication by a complex number of modulus one) a unitary or anti-unitary operator  $U$  satisfying

$$\alpha(x) = UxU^*, \quad \forall x \in \mathcal{B}(\mathcal{H}). \quad (1)$$

It can be further shown that if  $\mathbb{R} \ni t \mapsto \alpha_t \in \text{aut}(\mathcal{B}(\mathcal{H}))$  be a group of  $*$ -preserving continuous automorphisms, then

one has a unique one-parameter strongly continuous group of unitaries  $U_t$  such that

$$\alpha_t(x) = U_t x U_t^*, \quad \forall x \in \mathcal{B}(\mathcal{H}), t \in \mathbb{R}. \quad (2)$$

The reader can find details on these materials in ref. 4.

From the point of view of physical applications, the automorphisms relate to the dynamical evolution of either an isolated quantum system or a quantum system in equilibrium with its environment, evolving in quasi-isolated fashion. However, in situations in which the physical systems is far removed from equilibrium, one expects loss of information and the automorphism group is no longer appropriate, and furthermore, one notes that, in such a case the evolution is likely to be not reversible, thereby necessitating the study of the class of the semigroup of  $*$ -endomorphisms of  $\mathcal{B}(\mathcal{H})$  or more generally of an arbitrary  $C^*$  or von Neumann algebra.

At this point, there are two possibilities for moving forward: (i) study the semigroup of  $*$ -endomorphisms of the algebra concerned, (ii) go into more details of the evolution for non-equilibrium systems, viz. contractivity and complete positivity at each instant of time and construct a model (often stochastic) to realize such evolutions. This article will restrict itself to the second possibility and try to describe briefly the progress in this direction, with particular emphasis on the contributions over the last two decades by the school at the Delhi Centre of the Indian Statistical Institute, the details of which can be found in refs 4–6.

For the purposes of this article, we shall assume some knowledge of the basic ideas about  $C^*$  and von Neumann algebra and their representations. The reader is referred to the book of Takesaki<sup>2</sup> for the general theory and to the chapter 2 of ref. 6 for the portions relevant to our discussions.

The most important notion in this area is that of a completely positive (CP) map (and more generally that of a semigroup of CP-maps) acting on a suitable  $C^*$  or von Neumann algebra  $\mathcal{A}$ . A linear map  $T: \mathcal{A} \rightarrow \mathcal{A}$  is said to be completely positive if for any  $n \in \mathbb{N}$ ,

$$\sum_{i,j=1}^n y_i^* T(x_i^* x_j) y_j \geq 0, \quad \forall \{x_i, y_i\} \in \mathcal{A}. \quad (3)$$

It is clear that every  $*$ -automorphism (or even every  $*$ -homomorphism) of  $\mathcal{A}$  is a CP-map. But there exist more general class of CP-maps. In fact the following theorem describes the structure of every CP-map on  $\mathcal{A}$ .

**Theorem 1.1** (Stinespring). A linear map  $T: \mathcal{A} \rightarrow \mathcal{A}$  ( $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ ) is CP if and only if there is a triple  $(\mathcal{K}, \pi, V)$  consisting of a Hilbert space  $\mathcal{K}$ , a unital  $*$ -homomorphism  $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$  and  $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $T(x) = V^* \pi(x) V$ ,  $\forall x \in \mathcal{A}$ , and  $\{\pi(x) V u \mid x \in \mathcal{A}, u \in \mathcal{H}\}$  is total in  $\mathcal{K}$ . Such a triple associated with  $T$  is unique in the sense that if  $(\mathcal{K}', \pi', V')$  be another such triple, then there is a unitary operator  $\Gamma: \mathcal{K} \rightarrow \mathcal{K}'$  such that  $\pi'(x) = \Gamma \pi(x) \Gamma^*$  and  $V' = \Gamma V$ . Furthermore, if  $\mathcal{A}$  is a von Neumann algebra and  $T$  is normal, then  $\pi$  can be chosen to be normal.

Here normality of a map  $T: \mathcal{A} \rightarrow \mathcal{A}$  means the property that if there exists a net of positive elements  $\{x_\alpha\}$  of  $\mathcal{A}$  increasing (in strong operator topology) to  $x$  in  $\mathcal{A}$ , then the net  $\{T(x_\alpha)\}$  (which is necessarily positive) also increases to  $T(x)$ . The above theorem of Stinespring says that every CP-map is a  $*$ -homomorphism followed by a general conjugation.

Since the normal  $*$ -representations (equivalently,  $*$ -homomorphisms)  $\pi$  of a von Neumann algebra  $\mathcal{A}$  (in particular of  $\mathcal{B}(\mathcal{H})$ ) has a simple structure, viz.

$$\pi(x) = \Omega^*(x \otimes I_k) \Omega, \quad (4)$$

where  $k$  is a Hilbert space (which can be chosen to be separable whenever  $\mathcal{H}$  is)  $\Omega: \mathcal{K} \rightarrow \mathcal{H} \otimes k$  is a partial isometry such that the projection  $\Omega \Omega^*$  commutes with  $x \otimes I_k$  for all  $x \in \mathcal{A}$ , one can easily arrive at the explicit structure of all normal CP-maps on a von Neumann algebra  $\mathcal{A}$ .

**Corollary 1.2** (Kraus). Given any normal unital CP map  $T: \mathcal{A} \rightarrow \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  ( $\mathcal{A}$  a von Neumann algebra and  $\mathcal{H}$  is separable), there exist a sequence  $\{R_n\} \in \mathcal{B}(\mathcal{H})$  s.t.

$$T(x) = \sum_n R_n^* x R_n \quad (5)$$

$$I = \sum_n R_n^* R_n,$$

where the sums on the right-hand side converge strongly.

The next question that arises naturally is the following. Instead of a single CP-map on  $\mathcal{A}$ , if we are given a semigroup of CP maps on  $\mathcal{A}$ , what kind of structure does it have? This question aside from being of mathematical importance has the possibility of application in many physical problems. As we have mentioned earlier, the dynamical evolution of a physical system dictates a semigroup of maps (which becomes a group of maps if the evolution is reversible) and this semigroup is in general believed to be a family of CP-maps on the algebra of ob-

servables of the physical system (again in the special case of reversible quasiequilibrium evolution, this reduces to a group of automorphisms of the algebra).

More precisely, a quantum dynamical semigroup (QDS) on a  $C^*$  or von Neumann algebra  $\mathcal{A}$  is a contractive semigroup  $T_t$  on  $\mathcal{A}$  such that the map  $T_t$  is a normal CP-map on  $\mathcal{A}$  for each  $t \in \mathbb{R}_+$ . Furthermore, it is said to be conservative if  $T_t(I) = I$ ,  $\forall t \geq 0$ .

There is another way of looking at a QDS, i.e. when the algebra on which it acts is commutative or say an algebra of functions. As a simple example, consider a finite probability space  $S = \{1, 2, \dots, n\}$  with probability distribution given by the vector  $P = \{p_1, p_2, \dots, p_n\}$  on it. Let  $f$  be real-valued random variable, i.e.  $f: S \rightarrow \mathbb{R}$  and let  $T = (t_{ij})_{i,j=1}^n$  be a stochastic (Markov) matrix such that  $t_{ij} \geq 0$  and  $\sum_{j=1}^n t_{ij} = 1$ . Then one can associate a (discrete) evolution on the functions as  $(Tf)(i) = \sum_{j=1}^n t_{ij} f(j)$  and note that  $T$  is a positive map, i.e. maps positive functions to itself and preserves identity and that if we associate with each probability vector  $p$  (in a one-to-one fashion) with the dual  $\phi_p$  of the algebra of functions by setting  $\phi_p(f) = \sum_{i=1}^n p_i f(i)$ , then there is a dual action given by  $(T^* \phi_p)(\chi_k) = \sum_{i=1}^n p_i t_{ik}$ , where  $\chi_k$  is the characteristic function of the singleton  $\{k\}$ . Since for abelian algebras, positivity and complete positivity are equivalent, the discrete semigroup  $(T^n)_{n \geq 0}$  provides an example of a dynamical semigroup associated with a classical probabilistic set-up. In fact, it will be the endeavour of this article to show that in general this is a typical scenario for a QDS, at least when the semigroup has a bounded generator.

## Fock space, Weyl operators and quantum stochastic integration

First, we recall some well-known facts about Fock spaces. For a Hilbert space  $\mathcal{H}$  and positive integer  $n$ , let  $\mathcal{H}_n \equiv \mathcal{H}^{\otimes n}$  denote the  $n$ -fold tensor product of  $\mathcal{H}$ , and  $\mathcal{H}_0$  denotes the one-dimensional Hilbert space  $\mathbb{C}$ . The free Fock space  $\Gamma^f(\mathcal{H})$  is defined as

$$\Gamma^f(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The distinguished vector  $\Omega \equiv 1 \oplus 0 \oplus 0 \oplus \dots$  is called the vacuum. For two Hilbert spaces  $\mathcal{H}, \mathcal{K}$  and a contraction  $T: \mathcal{H} \rightarrow \mathcal{K}$ , we denote by  $T_n$  the  $n$ -fold tensor product of  $T$  and set  $T_0 = 1$ . Let us define  $\Gamma^f(T) = \bigoplus_{n=0}^{\infty} T_n: \Gamma^f(\mathcal{H}) \rightarrow \Gamma^f(\mathcal{K})$ . Then, it is possible to verify the following.

**Lemma 2.1.**  $\Gamma^f$  is a functor on the category whose objects are Hilbert spaces and morphisms are contractions, that is,  $\Gamma^f(ST) \equiv \Gamma^f(S) \Gamma^f(T)$ ,  $\Gamma^f(I) = I$ . Furthermore,  $\Gamma^f(0)$  is the projection on the Fock vacuum vector and  $\Gamma^f(T^*) = (\Gamma^f(T))^*$ .

Let  $\mathcal{H}_n^s$  and  $\mathcal{H}_n^a$  denote respectively, the symmetric and antisymmetric  $n$ -fold tensor products of  $\mathcal{H}$  for any posi-

tive integer  $n$ , and  $\mathcal{H}_0^s = \mathcal{H}_0^a = \mathcal{H}_0$ . Then the symmetric (or Boson) and antisymmetric (or Fermion) Fock spaces over  $\mathcal{H}$ , denoted respectively by  $\Gamma^s(\mathcal{H})$  and  $\Gamma^a(\mathcal{H})$ , are defined as

$$\Gamma^s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^s.$$

$$\Gamma^a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^a.$$

We shall be mostly concerned with the symmetric Fock spaces in the present work, and hence for simplicity of notation, we shall use the notation  $\Gamma(\mathcal{H})$  for the symmetric Fock space. Let us mention the basic factorization property of  $\Gamma(\mathcal{H})$ .

**Theorem 2.2.** Consider the map  $\mathcal{H} \ni u \mapsto e(u) \in \Gamma(\mathcal{H})$  given by  $e(u) = \bigoplus_{n=0}^{\infty} (n!)^{-1/2} u^{\otimes n}$ , where  $u^{\otimes n}$  is the  $n$ -fold tensor product of  $u$  for positive  $n$  and  $u^{\otimes 0} = 1$ . Then the map  $e(\cdot)$  is the minimal Kolmogorov decomposition (see refs 4 and 6 for the details) for the positive definite kernel  $\mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$  given by  $\{u, v\} \mapsto \exp(\langle u, v \rangle)$ . Furthermore,  $\{e(u) : u \in \mathcal{H}\}$  is a linearly independent total set of vectors in  $\Gamma(\mathcal{H})$ .

*Proof.* That  $e(\cdot)$  is a Kolmogorov decomposition for the above-mentioned kernel is verified by the relation  $\langle e(u), e(v) \rangle = \exp(\langle u, v \rangle)$ . Furthermore, the relation

$$\frac{d^n}{dt^n} e(tu)|_{t=0} = (n!)^{1/2} u^{\otimes n} \quad (6)$$

shows that for every  $u \in \mathcal{H}$ ,  $u^{\otimes n}$  belongs to the closed linear span of  $e(u)$ . Since the vectors of the form  $u^{\otimes n}$  where  $n$  varies over  $\{0, 1, 2, \dots\}$  are total in  $\Gamma(\mathcal{H})$ , the assertion about minimality follows. To prove the linear independence, suppose that  $u_1, u_2, \dots, u_n$  are distinct vectors in  $\mathcal{H}$  and  $z_1, \dots, z_n$  are complex numbers such that  $\sum_{j=1}^n z_j e(u_j) = 0$ . Then we have, for all  $t \in \mathbb{R}$ ,  $\sum_{j=1}^n z_j \times \exp(t\langle u_j, v \rangle) = 0$  for all  $v \in \mathcal{H}$ . Since  $u_1, u_2, \dots, u_n$  are distinct, there exists  $v \in \mathcal{H}$  such that the scalars  $\langle u_j, v \rangle$  are distinct and hence the functions  $\{e^{t\langle u_j, v \rangle}\}$  are linearly independent, which implies that  $z_j = 0$  for all  $j$ .

**Corollary 2.3.** For any dense subset  $S$  of  $\mathcal{H}$ , the set  $\{e(u) : u \in S\}$  is total in  $\Gamma(\mathcal{H})$ .

**Corollary 2.4.** There is a natural identification of  $\Gamma(\mathcal{H} \oplus \mathcal{K})$  with  $\Gamma(\mathcal{H}) \otimes \Gamma(\mathcal{K})$  under which  $e(u \oplus v) \mapsto e(u) \otimes e(v)$ .

For contraction  $C$  on  $\mathcal{H}$ , we define the second quantization  $\Gamma(C)$  on  $\Gamma(\mathcal{H})$  by

$$\Gamma(C)e(u) = e(Cu).$$

We observe the following.

**Lemma 2.5.**  $\Gamma(C)$  admits an extension to  $\Gamma(\mathcal{H})$  and the extension, denoted also by  $\Gamma(C)$ , is a contraction. Moreover, if  $C$  is isometry (respectively unitary), then so is  $\Gamma(C)$ .

*Proof.* For  $\alpha_i \in \mathbb{C}$ , vectors  $u_i \in \mathcal{H}$  ( $i = 1, \dots, n$ ), we have

$$\begin{aligned} & \left\| \Gamma(C) \left( \sum_{i=1}^n \alpha_i e(u_i) \right) \right\|^2 \\ &= \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \exp(\langle u_i, C^* C u_j \rangle) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \langle u_i, C^* C u_j \rangle^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \exp(\langle u_i^{\otimes m}, (C^* C)^{\otimes m} u_j^{\otimes m} \rangle) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \langle \bar{u}_m, (C^* C)^{\otimes m} \bar{u}_m \rangle. \end{aligned}$$

where we have denoted the  $m$ -fold tensor product of a vector or an operator by the symbol  $\otimes^m$  (with  $v^{\otimes 0} := 1 \in \mathbb{C}$ ,  $(C^* C)^{\otimes 0} := 1$ ), and by  $\bar{u}_m$  the vector  $\sum_{i=1}^n \alpha_i u_i^{\otimes m}$ . Since  $(C^* C)^{\otimes m}$  is a positive contraction for every  $m$ , we have

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{1}{m!} \langle \bar{u}_m, (C^* C)^{\otimes m} \bar{u}_m \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \langle \bar{u}_m, \bar{u}_m \rangle \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \langle u_i, u_j \rangle^m = \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \exp(\langle u_i, u_j \rangle) \\ &= \left\| \sum_{i=1}^n \alpha_i e(u_i) \right\|^2. \end{aligned}$$

This completes the proof that  $\Gamma(C)$  extends to a contraction, since the linear span of exponential vectors is a dense subset of  $\Gamma(\mathcal{H})$ . It is straightforward to see that  $\Gamma(C)$  is an isometry (respectively unitary) whenever  $C$  is so.

For  $u \in \mathcal{H}$  and unitary operator  $U$  in  $\mathcal{H}$ , we define the Weyl operators  $W(u, U)$  by setting

$$W(u, U)e(v) = \exp \left( -\frac{1}{2} \|u\|^2 - \langle u, Uv \rangle \right) e(u + Uv).$$

It is known that the von Neumann algebra generated by the family  $\{W(u, I) : u \in S\}$  is the whole of  $B(\Gamma(\mathcal{H}))$  whenever  $S$  is a dense subspace of  $\mathcal{H}$ . One can also verify the following properties of the Weyl operators: (i) the correspondence  $(u, U) \mapsto W(u, U)$  is strongly continuous, and (ii) they satisfy the Weyl commutation rule:

$$W(u_1, U_1) W(u_2, U_2) = \exp(-i \operatorname{Im} \langle u_1, U_1 u_2 \rangle) \times W(u_1 + U_1 u_2, U_1 U_2).$$

we refer the reader to ref. 4 for a further detailed description of Fock spaces and related topics with applications to quantum probability.

Let  $h$  and  $k_0$  be two separable Hilbert spaces, and we consider first the tensor product Hilbert space  $\mathcal{H} = h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$ . This Hilbert space admits the following continuous tensor product decomposition: for  $0 \leq s < t < \infty$ ,

$$\mathcal{H} \simeq h \otimes \Gamma(L^2([0, s], k_0)) \otimes \Gamma(L^2([s, t], k_0)) \otimes \Gamma(L^2([t, \infty], k_0)) \equiv (h \otimes \Gamma_s) \otimes \Gamma_t^s \otimes \Gamma^t \quad (7)$$

given by the decomposition of the exponential vectors  $e(f) \simeq e(f\chi_{[0,s]}) \otimes e(f\chi_{[s,t]}) \otimes e(f\chi_{[t,\infty]})$ , where  $\simeq$  means the isomorphism of Hilbert spaces. Let  $\{e_j\}_{j \geq 1}$  be an orthonormal basis of  $k_0$ . We set for an interval  $\Delta \subseteq \mathbb{R}_+$ ;  $j = 1, 2, 3, \dots, f \in L^2(\mathbb{R}_+, k_0)$

$$A_j(\Delta)e(f) = \langle \chi_\Delta e_j, f \rangle e(f),$$

$$A_j^+(\Delta)e(f) = -i \frac{d}{dt} e(f + t\chi_\Delta e_j)|_{t=0},$$

$$\Lambda_k^j(\Delta)e(f) = -i \frac{d}{dt} e(\exp[it\chi_\Delta |e_j \rangle \langle e_k|] f)|_{t=0},$$

where  $|e_j \rangle \langle e_k|$  is the associated rank-one operator in  $k_0$ . These three (unbounded) operator-increments give rise to the three fundamental quantum stochastic integrators called annihilation, creation and conservation increments. Here it is worth mentioning the relationship between this picture and the classical probabilistic picture. If  $\omega \in C_0(\mathbb{R}_+)$ , the linear space of continuous real-valued functions on  $\mathbb{R}_+$  vanishing at 0, equipped with the topology of uniform convergence on compact subsets of  $\mathbb{R}_+$ , then following Wiener, one constructs a measure  $\mathbb{P}$  on  $C_0(\mathbb{R})$  such that with respect to this measure the increments  $\{[\omega(t) - \omega(s)]: 0 \leq s < t < \infty\}$  constitute independent increment Gaussian random variables (called the standard Brownian motion) with mean 0 and variance  $(t - s)$ .

In such a set-up, the Hilbert space  $L^2(\mathbb{P})$  is isometrically isomorphic with the (symmetric) Fock space  $\Gamma(L^2(\mathbb{R}_+))$ , the isomorphism being given by the association: for  $f \in L^2(\mathbb{R}_+)$ , and  $0 \leq s < t < \infty$ ,

$$\exp \left[ \int_s^t f(\tau) d\omega(\tau) - \frac{1}{2} \int_s^t f(\tau)^2 d\tau \right] \leftrightarrow e(f\chi_{[s,t]}),$$

such that

$$\begin{aligned} & \int d\mathbb{P}(\omega) \left[ \int_0^\infty f(s) d\omega(s) - \frac{1}{2} \int_0^\infty f(s)^2 ds \right] \\ & \times \exp \left[ \int_0^\infty g(s) d\omega(s) - \frac{1}{2} \int_0^\infty g(s)^2 ds \right] \\ & = \exp(\langle f, g \rangle_{L^2(\mathbb{R}_+)}) = \langle e(f), e(g) \rangle_\Gamma. \end{aligned}$$

Furthermore, under the same isomorphism the operator of multiplication by  $\{\omega(s)\}$  goes over to the (unbounded) self-adjoint operator  $Q(s) = A(s) + A^+(s) \equiv A([0, s]) + A^+([0, s])$ .

Next, we want to define quantum stochastic integration for a class of integrands. We call a function  $\mathbb{R}_+ \ni t \xrightarrow{L} L(t)$  an adapted process with respect to the ‘filtration’ given by the continuous tensor-product decomposition of the Fock space, if  $L(t)$  is of the form  $L_t \otimes I_{[t,\infty]}$  for the Fock space factorization:  $(h \otimes \Gamma_t) \otimes \Gamma^t$  for every  $t \geq 0$ . At this point, it is useful to introduce a compact notation. The Greek indices  $\alpha, \beta$  will run over  $(0, 1, 2, \dots) \equiv \{0\} \cup \mathbb{N}$  while  $j, k \in \mathbb{N}$ . With that convention, we set  $\Lambda_0^0(\Delta) = |\Delta|$ , the Lebesgue measure of  $\Delta$ ,  $\Lambda_0^j(\Delta) = A_j(\Delta)$ ,  $\Lambda_j^0(\Delta) = A_j^+(\Delta)$ .

A  $(D, D_0)$ -adapted process  $L$  is called measurable, given the dense sets  $D \subseteq h$  and  $D_0 \subseteq L^2(\mathbb{R}_+, k_0)$ , if the map  $t \mapsto L(t)ue(f)$  (for every  $u \in D$  and  $f \in D_0$ ) is Borel measurable. Such a measurable process is called (quantum)-stochastically integrable if there exists a sequence  $\{L_n(t)\}_{t \geq 0}$  of simple  $(D, D_0)$ -adapted processes such that

$$\lim_{n \rightarrow \infty} \int_0^t \|L_n(s)ue(f) - L(s)ue(f)\|^2 (1 + \|f(s)\|^2) ds = 0$$

$$\text{for all } u \in D, f \in D_0 \subseteq L^2(\mathbb{R}_+, k_0), t \geq 0.$$

Then the following lemma give the tools to compute the stochastic integrals and their product (for proofs, see refs 4 and 6).

**Lemma 2.6.** (i) Let  $\{L(t)\}_{t \geq 0}$  be  $(D, D_0)$ -adapted measurable and stochastically integrable process. Then for  $v, u \in D \subseteq h, f, g \in D_0 \subseteq \Gamma, t \geq 0$ ,

$$\begin{aligned} & \left\langle ue(f), \int_0^t L(s) \begin{Bmatrix} dA_j(s) \\ dA_j^+(s) \\ d\Lambda_k^j(s) \end{Bmatrix} ve(g) \right\rangle \\ & = \int_0^t ds \langle ue(f), L(s)ve(g) \rangle \begin{Bmatrix} g_j(s) \\ \bar{f}_j(s) \\ \bar{f}_j(s)g_k(s) \end{Bmatrix} \text{ respectively.} \end{aligned}$$

(ii) Let  $\{L_1(t)\}_{t \geq 0}$  and  $\{L_2(t)\}_{t \geq 0}$  be two  $(D, D_0)$ -adapted, measurable and stochastically integrable processes. Then for  $u, v \in D, f, g \in D_0$  and  $t \geq 0$ ,

$$\left\langle \int_0^t L_1(s) \begin{Bmatrix} dA_j(s) \\ dA_j^+(s) \\ d\Lambda_k^j(s) \end{Bmatrix} ue(f), \int_0^t L_2(s) \begin{Bmatrix} dA_l(s) \\ dA_l^+(s) \\ d\Lambda_m^l(s) \end{Bmatrix} ve(g) \right\rangle$$

$$\begin{aligned}
&= \int_0^t ds \left\langle L_1(s) u e(f), \int_0^s L_2(x) \begin{Bmatrix} dA_l^+(x) \\ d\Lambda_m^l(x) \end{Bmatrix} v e(g) \right\rangle \\
&\quad \times \begin{Bmatrix} \bar{f}_j(s) \\ g_j(s) \\ \bar{f}_j(s) g_k(s) \end{Bmatrix} \\
&+ \int_0^t ds \left\langle \int_0^s L_1(x) \begin{Bmatrix} dA_j^+(x) \\ d\Lambda_k^j(x) \end{Bmatrix} u e(f), L_2(s) v e(g) \right\rangle \\
&\quad \times \begin{Bmatrix} g_l(s) \\ \bar{f}_l(s) \\ \bar{f}_l(s) g_m(s) \end{Bmatrix} + \int_0^t \langle L_1(s) u e(f), L_2(s) v e(g) \rangle \\
&\quad \times \begin{Bmatrix} 0 & 0 & 0 \\ 0 & \delta_l^j & \delta_m^j g_l(s) \\ 0 & \delta_l^k \bar{f}_j(s) & \delta_m^k \bar{f}_l(s) g_j(s) \end{Bmatrix} ds
\end{aligned}$$

respectively. The third term on the right-hand side of the above expression reflects what are called the ‘quantum Ito correction terms’.

Next we consider quantum stochastic differential equations (QSDE) with constant  $\mathcal{B}(h)$ -valued or a class of unbounded operator in  $h$ -valued coefficients and their solutions.

Consider the QSDE in  $h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$  with initial value as given below:

$$\begin{aligned}
V_t &= I_{h \otimes \Gamma} + \int_0^t V_s \left\{ \sum_{j,k} (\Omega_j^{k*} - \delta_k^j) d\Lambda_j^k(s) \right. \\
&\quad \left. + \sum_j \left( (R_j dA_j^*(s) - \sum_{j,k} R_k^* \Omega_k^{j*} dA_j(s)) \right) \right. \\
&\quad \left. + \left( iH - \frac{1}{2} \sum_j R_j^* R_j \right) ds \right\} \equiv I + \sum_{\alpha, \beta \geq 0} \int_0^t V_s \sum L_\beta^\alpha d\Lambda_\alpha^\beta(s), \quad (8)
\end{aligned}$$

where

$$L_\alpha^\beta = \begin{cases} \Omega_j^{k*} - \delta_k^j; & \alpha = j, \beta = k \geq 1 \\ R_j; & \alpha = j \geq 1, \beta = 0 \\ -\sum_k R_k^* \Omega_k^{j*}; & \alpha = 0, \beta = j \geq 1 \\ \left( iH - \frac{1}{2} \sum_j R_j^* R_j \right); & \alpha = \beta = 0, \end{cases} \quad (9)$$

and  $\{\Lambda_\alpha^\beta(ds)\}$  has already been introduced earlier, with  $R_j \in \mathcal{B}(\mathcal{H})$  for  $j \geq 1$ ,  $H$  a bounded self-adjoint operator in

$h$ ,  $\Omega \equiv ((\Omega_k^l))_{j,k \geq 1} : h \otimes k_0 \rightarrow h \otimes k_0$  is a unitary operator. Then we have the following theorem<sup>4,6</sup>.

**Theorem 2.7.** The QSDE (8) has a unique unitary operator-valued  $(h, L^2(\mathbb{R}_+, k_0))$ -adapted measurable solution  $V_t$ .

### Solution of a class of QSDE with coefficients which are bounded maps on a von Neumann algebra

We start with a QDS  $\{T_t\}_{t \geq 0}$  on a von Neumann algebra  $\mathcal{A}$  (acting in a Hilbert space  $h$ ) with bounded generator  $\mathcal{L}$  which then turns out to be also a normal map on  $\mathcal{A}$ . As will be shown in the next section, in such a case there is a canonically associated triplet  $(k_0, \delta, \sigma)$  with  $k_0$  a Hilbert space (which can be chosen to be separable if  $h$  is separable),  $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0)$  and  $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$ , where  $\mathcal{B}(X, Y)$  stands for the normed linear space of the bounded linear maps from  $X$  to  $Y$ . These satisfy the following:

(A1)  $\sigma(x) = \Sigma^*(x \otimes I_{k_0})\Sigma - x \otimes I_{k_0} \equiv \pi(x) - x \otimes I_{k_0}$  for  $x \in \mathcal{A}$ , where  $\Sigma$  is a partial isometry in  $h \otimes k_0$  such that  $\pi$  is a  $*$ -representation of  $\mathcal{A}$  in  $h \otimes k_0$ ;

(A2)  $\delta(x) = Rx - \pi(x)R$ , where  $R \in \mathcal{B}(h, h \otimes k_0)$  so that  $\delta$  is  $\pi$ -derivation, i.e.

$$\delta(xy) = \delta(x)y + \pi(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A},$$

(A3)  $\mathcal{L}(x) = R^*\pi(x)R + lx + x l^*$ , where  $l \in \mathcal{A}$  with the condition that  $\mathcal{L}(I) = 0$ , so that  $\mathcal{L}$  satisfies the second-order cocycle relations with  $\delta$  as coboundary, i.e.

$$\mathcal{L}(x^*y) - x^*\mathcal{L}(y) - \mathcal{L}(x)^*y = \delta(x)^*\delta(y) \quad \text{for all } x, y \in \mathcal{A}.$$

The triple of the maps  $(\mathcal{L}, \delta, \sigma)$  on  $\mathcal{A}$  are called the structure maps and often one associates a structure matrix with it:

$$\Theta(x) = \begin{bmatrix} \mathcal{L}(x) & \delta^+(x) \\ \delta(x) & \Sigma(x) \end{bmatrix} \quad \text{for } x \in \mathcal{A}, \text{ where}$$

$$\delta^+(x) = \delta(x^*)^* \text{ so that } \Theta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \hat{k}_0)$$

with  $\hat{k}_0 = \mathbb{C} \oplus k_0$ . If  $h$  and hence  $k_0$  or  $\hat{k}_0$  is separable, then by choosing an orthonormal basis  $\{e_0 \equiv 1, e_1, e_2, \dots, e_N\}$  ( $N$  could be finite or infinite) of  $k_0$ , we can write  $\Theta$  in this basis as a matrix  $((\Theta_{\beta\alpha}^j))$  of maps from  $\mathcal{A}$  into  $\mathcal{A}$  as follows:

$$\Theta_0^0(x) = \mathcal{L}(x), \Theta_j^0(x) = \delta_j(x) \equiv \langle \delta(x), e_j \rangle \in \mathcal{A},$$

$$\Theta_0^j(x) = \delta_j^+(x) = \delta_j(x^*)^* \in \mathcal{A}, \Theta_k^j(x) = \pi_k^j(x) - \delta_k^j x,$$

for  $x \in \mathcal{A}$  and  $j, k = 1, 2, 3, \dots$ . Thus it is easy to verify using (A1)–(A3) that the structure maps satisfy the following structure relation:

$$\Theta_{\beta}^{\alpha}(x, y) = \Theta_{\beta}^{\alpha}(x)y + x\Theta_{\beta}^{\alpha}(y) + \sum_{j=1}^N \Theta_j^{\alpha}(x)\Theta_{\beta}^j(y). \quad (10)$$

Then with respect to this basis, consider the following QSDE (or equivalently, the integral equation) for the map  $j_t: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma)$  for  $t \geq 0$

$$j_t(x) = x \otimes I_{\Gamma} + \sum_{\alpha, \beta} \int_0^t j_s(\Theta_{\beta}^{\alpha}(x)) \Lambda_{\alpha}^{\beta}(ds), \quad (11)$$

where  $\{\Lambda_{\beta}^{\alpha}(t)\}$  are the (unbounded) operator processes described in the previous section. Then the principal result in this section is the following theorem (see refs 6–8).

**Theorem 3.1.** Let  $\mathcal{A}$  be a von Neumann algebra of operator acting in a separable Hilbert space  $h$  and let  $((\Theta_{\beta}^{\alpha}))$  be the matrix of bounded structure maps on  $\mathcal{A}$  satisfying (A1)–(A3) or equivalently eq. (10). Then the QSDE (11) has a unique solution  $j_t(x)$  such that

- (i) for every  $t \geq 0$ ,  $j_t: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma)$  is a  $*$ -homomorphism,
- (ii)  $\|j_t(x)\| \leq \|x\|$ ,  $t \mapsto j_t(x)$  is strongly continuous,
- (iii)  $\mathcal{A} \ni x \mapsto j_t(x) \in \mathcal{A} \otimes \mathcal{B}(\Gamma)$  is normal.
- (iv)  $\langle u, \mathbb{E}(j_t(x))v \rangle = \langle u \otimes e(0), j_t(x)v \otimes e(0) \rangle = \langle u, T_t(x)v \rangle$ , where  $\{T_t\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $\mathcal{A}$  with the original generator  $\mathcal{L}$ , i.e.  $T_t$  is the QDS we had started with.

A proof of this can be found in refs 6 and 7. In effect, the theorem establishes the existence of  $*$ -homomorphic stochastic flow (or diffusion) over a von Neumann algebra  $\mathcal{A}$ , satisfying the QSDE such that its (vacuum) expectation gives back the original QDS. As a special case, one can recover for example, simple classical diffusion in  $\mathbb{R}^d$ , provided one ignores the fact that all the structure maps are ‘unbounded’ in this case. The von Neumann algebra  $\mathcal{A} = L^{\infty}(\mathbb{R}^d)$ ,  $\Theta_0^0(\varphi)(x) = -(1/2)(\Delta\varphi)(x)$  for  $\varphi \in \mathcal{A}_{\infty} \equiv BC^{\infty}(\mathbb{R}^d)$  (the  $*$ -subalgebra of  $\mathcal{A}$  consisting of bounded smooth function on  $\mathbb{R}^d$ ),  $\Theta_k^j = 0$  for  $j, k = 1, 2, 3, \dots, d$  and  $\Theta_j^j(\varphi) = \Theta_0^j(\varphi) = i\partial_j\varphi$  for  $j = 1, 2, \dots, d$  so that the QSDE in this example reduces to

$$j_t(\varphi) = \varphi \otimes I_{\Gamma(L^2(\mathbb{R}_+, \mathbb{R}^d))} + \int_0^t j_s \left( -\frac{1}{2} \Delta \varphi \right) ds + \sum_{j=1}^d \int_0^t j_s(i\partial_j \varphi) d\theta_j(s), \quad (12)$$

where  $Q_j(s) = \mathcal{A}_j(s) + \mathcal{A}_j^+(s) \simeq$  multiplication by the  $j$ th component of the  $\mathbb{R}^d$ -valued standard Brownian motion

$\{w(s)\}$  and  $\varphi \in \mathcal{A}_{\infty}$ . It is also clear what the solution of eq. (12) is going to be like, in fact:  $j_t(\varphi)(x) = \varphi(x + w(t))$  which belongs to  $\mathcal{A}$  for almost all  $w$ , i.e.  $j_t: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{P}))$  is a  $*$ -homomorphism of  $\mathcal{A}$ .

Thus the QSDE (11) is a generalization of the classical diffusion equation like (12), albeit with bounded structure maps. The case of unbounded structure maps, in its full generality, is far too complex; however, for a class of QDS’s there is a theory which we shall discuss in the last section.

It is worth noting that if we take the (classical) expectation of  $j_t(\varphi)$ , i.e.  $T_t(\varphi) = \mathbb{E}j_t(\varphi) = \int \varphi(x + w(t)) \mathbb{P}(dw)$ , then  $T_t$  is a  $C_0$ -semigroup on  $\mathcal{A}$  and its (unbounded) generator is  $-\frac{1}{2}\Delta$ .

### Existence of structure maps compatible with a given QDS

The question that needs to be addressed now is the following: is there a set of structure maps  $(\Theta_{\beta}^{\alpha})$  or equivalently, the triple  $(k_0, \delta, \sigma)$  mentioned in the previous section which are canonically associated with the given QDS on a von Neumann algebra. The answer is given in the next theorem.

**Theorem 4.1.** Let  $\{T_t\}_{t \geq 0}$  be a norm-continuous conservative QDS, i.e.  $T_t(I) = I$ ,  $\forall t$  on a von Neumann algebra  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  ( $h$  separable) with generator  $\mathcal{L}$ . Then there exists a separable Hilbert space  $k_0$ , a normal  $*$ -representation  $\pi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(k_0)$  and a  $\pi$ -derivation  $\delta: \mathcal{A} \rightarrow \mathcal{A} \otimes k_0$  such that the properties (A1)–(A3) in the previous section are satisfied.

A proof of the result can be found in ref. 6. It should be noted that there is no uniqueness statement here. In fact, this problem (sometimes called the ‘stochastic dilation problem’ of a QDS) has many possible solutions, not the least of which is due to the fact that the dimension of the Hilbert space  $k_0$  can be quite arbitrary. It may be possible to give a minimal dimension on the basis of some further hypothesis on the QDS, but this is not known at present.

This section, read along with the previous one, gives the following fairly satisfactory picture: Given a norm-continuous QDS  $\{T_t\}_{t \geq 0}$  on a von Neumann algebra  $\mathcal{A} \subseteq \mathcal{B}(h)$  with bounded generator  $\mathcal{L}$ , there exists a flow of  $*$ -homomorphisms  $\{j_t\}_{t \geq 0}$  from  $\mathcal{A}$  to  $\mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$  satisfying the QSDE (11) such that  $\mathbb{E}(j_t(x)) = T_t(x)$ ,  $\forall x \in \mathcal{A}$ . This solves the dilation problem completely (though non-uniquely) for the case of norm continuous QDS.

Another question may arise that when is this family of  $*$ -homomorphisms unitarily implemented, just as was the case with Wigner’s theorem, mentioned in the introduction. The next theorem answers this in the affirmative for the norm-continuous QDS.

As we have explained in the previous section and in the introduction,

$$\pi(x) = \Omega^*(x \otimes I_{k_0})\Omega \in \mathcal{A} \otimes \mathcal{B}(k_0),$$

$$\delta(x) = Rx - \pi(x)R \in \mathcal{A} \otimes k_0 \text{ with } R \in \mathcal{B}(h, h \otimes k_0)$$

and with  $\Omega$  a partial isometry in  $h \otimes k_0$  such that  $\Omega\Omega^*$  commute with  $x \otimes I_{k_0}$ ,  $\forall x \in \mathcal{A}$ .

Let us consider now the QSDE (8) in  $h \otimes (\Gamma(L^2(\mathbb{R}_+, k_0)))$ :

$$\begin{aligned} V_t &= I_{h \otimes \Gamma} + \int_0^t V_s \sum_{j,k=1}^{\infty} (\Omega_j^{k*} - \delta_k^j) d\Lambda_j^k(s) \\ &+ \int_0^t V_s \sum_{j=1}^{\infty} \left( R_j dA_j^+(s) - \sum_k R_k^* \Omega_k^{j*} dA_j(s) \right) \\ &+ \int_0^t V_s \left( iH - \frac{1}{2} \sum_{j=1}^{\infty} R_j^* R_j \right) ds \end{aligned}$$

where  $\{\Omega_j^k\}$  and  $\{R_j\}$  are the components of the operators  $\Omega$  and  $R$  in a basis of  $k_0$  as described earlier, except that  $\Omega$  is only a partial isometry here and not unitary.

Then we have the next result, the proof of which can be found in ref. 6.

**Theorem 4.2.** Let  $\{j_t\}_{t \geq 0}$  be the  $*$ -homomorphic flow on the von Neumann algebra  $\mathcal{A}$ , associated to a QDS  $T_t$  as described in theorem 3.1. Then the QSDE (13) in  $h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$  has a unique adapted contraction valued process. Furthermore,  $V_t$  is a partial isometry-valued process and the  $*$ -homomorphism  $j_t$  is implemented by  $V_t$ , i.e.  $j_t(x) = V_t(x \otimes I_{\Gamma})V_t^*$  with the projection-valued processes  $\hat{F}_t \equiv V_t V_t^*$  and  $G_t \equiv V_t^* V_t$  belonging to  $\mathcal{A} \otimes \mathcal{B}(\Gamma)$  and  $\mathcal{A}' \otimes \mathcal{B}(\Gamma)$  respectively.

In other words, the last two theorems together complete the picture for a QDS on a von Neumann algebra just as Wigner's theorem did for the group of  $*$ -automorphisms. The analogy is complete except for the fact that the dilation and its implementation are both stochastic in nature. Another feature of the theorem is worth noting, viz. while the map  $\pi$  in eq. (4) gives a  $*$ -homomorphism of the algebra  $\mathcal{A}$  in  $\mathcal{B}(h \otimes k_0)$ , the family of maps  $\{j_t\}_{t \geq 0}$  gives a flow of  $*$ -homomorphisms. Similarly, while the partial isometry  $\Omega$  is such that  $\Omega\Omega^* \in \mathcal{A}' \otimes \mathcal{B}(k_0)$ ,  $V_t^* V_t \in \mathcal{A}' \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+, k_0)))$ ,  $\forall t \geq 0$ . In other words,  $\pi \rightarrow j_t$  is a kind of 'exponentiation map'.

## QDS with unbounded generator

This problem is naturally more difficult to handle and one needs to have more 'smoothness structure', compatible

with the QDS, on the algebra on which the QDS acts. This is achieved by assuming that there is an action of a second countable Lie group  $G$  on the  $C^*$  or von Neumann algebra  $\mathcal{A} \subseteq \mathcal{B}(h)$ , i.e. there exists a map  $G \ni g \mapsto \alpha_g \in \text{aut}(\mathcal{A})$ , the group of  $*$ -automorphisms of  $\mathcal{A}$ , which is furthermore assumed to be continuous with respect to the strong operator topology induced by  $\mathcal{B}(h)$ . It is furthermore assumed that the QDS  $\{T_t\}_{t \geq 0}$  is covariant under the above action, i.e.  $T_t(\alpha_g(x)) = \alpha_g(T_t(x))$  for all  $x \in \mathcal{A}$ ,  $g \in G$  and  $t \geq 0$ . If we set  $\mathcal{A}_{\infty}$  to be the subset of  $\mathcal{A}$  consisting of all elements  $x$  such that  $g \mapsto \alpha_g(x)$  is arbitrarily often differentiable with respect to the norm topology, then it can be shown that  $\alpha_g(\mathcal{A}_{\infty}) \subseteq \mathcal{A}_{\infty}$  and that  $\mathcal{A}_{\infty}$  is a dense  $*$ -subalgebra of  $\mathcal{A}$ .

If furthermore, there exists a densely defined, semi-finite, lower semi-continuous, faithful trace  $\tau$  on  $\mathcal{A}$ , invariant under  $\alpha_g$ ; then we can choose  $h = L^2(\tau)$ , the non-commutative  $L^2$ -space with respect to  $\tau$ , in which  $\mathcal{A}$  acts by left-multiplication. The invariance implies that  $\tau(\alpha_g(x^*y)) = \tau(x^*y)$  for  $x, y \in \mathcal{A}_{\tau} \equiv \{x \in \mathcal{A} | \tau(x^*x) < \infty\}$ , which is ultra-weakly dense in the von Neumann algebra generated by  $\mathcal{A}$  in  $\mathcal{B}(\mathcal{H})$ . Thus the action  $\alpha_g$  of the group in this case gives rise to a strongly continuous unitary representation in  $\{u_g\}_{g \in G}$  in  $h$ .

We denote by  $h_{\infty}$ , the subset of  $h$  consisting of vectors  $\xi$  such that  $g \mapsto u_g \xi$  is arbitrarily often differentiable with respect to the Hilbert space topology of  $h$ , and note that  $h_{\infty}$  is dense in  $h$ . Now we assume that the QDS  $\{T_t\}_{t \geq 0}$  is symmetric with respect to  $\tau$ , i.e.  $T_t(\mathcal{A}_{\tau}) \subseteq \mathcal{A}_{\tau}$  and  $\tau(T_t(x)y) = \tau(xT_t(y))$  for all  $x, y \in \mathcal{A}_{\tau}$ . In such a case, the QDS  $\{T_t\}_{t \geq 0}$  can be canonically extended to a  $C_0$ -semi-group of positive contractions on  $h$ . We have already denoted by  $\mathcal{L}$ , the generator of  $\{T_t\}_{t \geq 0}$  with respect to the weak operator topology of  $\mathcal{A}$ , and now we denote by  $\mathcal{L}_2$  the generator of  $\{T_t\}_{t \geq 0}$  in  $h$ , and note that  $\mathcal{L}_2$  is a negative self-adjoint operator in  $h$ . Then we have the following result whose proof can be found in refs 6 and 9.

**Theorem 5.1.** Let  $\mathcal{A}$  be a von Neumann algebra with a semifinite faithful lower semicontinuous trace  $\tau$  on it such that the QDS  $\{T_t\}_{t \geq 0}$  acting on  $\mathcal{A}$  is symmetric with respect to  $\tau$ . Furthermore assume that there is an action of a Lie group  $G$  on  $\mathcal{A}$  such that  $\tau$  is invariant and  $T_t$  is covariant under this action and such that  $\mathcal{A}_{\infty} \subseteq D(\mathcal{L})$  and  $h_{\infty} \subseteq D(\mathcal{L}_2)$ .

Then, (i) there exist a Hilbert space  $\mathcal{K}$  and a linear operator  $R: h \rightarrow \mathcal{K}$  with  $D(R) = \mathcal{A}_{\infty} \cap h_{\infty}$  such that  $\mathcal{L}_2(x) = -\frac{1}{2}R^*Rx$  for  $x \in D(\mathcal{L}_2)$ , and  $\mathcal{L}(x) = R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R$  for all  $x \in \mathcal{A}_{\infty} \cap h_{\infty}$ ;

(ii) If  $G$  is compact, then the assumption that  $\mathcal{A}_{\infty} \subseteq D(\mathcal{L})$  implies that  $h_{\infty} \subseteq D(\mathcal{L}_2)$ ;

(iii) There exists a Hilbert space  $k_1$ , a partial isometry  $\Sigma: \mathcal{K} \rightarrow h \otimes k_0$  (where  $k_0 = L^2(G) \otimes k_1$ ) such that  $\pi(x) = \Sigma^*(x \otimes I_{k_0})\Sigma$  and  $\hat{R} \equiv \Sigma R$  is covariant in the sense that  $(u_g \otimes v_g)\hat{R} = \hat{R}u_g$  on  $\mathcal{A}_{\infty} \cap h_{\infty}$  with  $v_g = L_g \otimes I_{k_1}$  on  $k_0, L_g$  denoting the left regular representation of  $G$  in  $L^2(G)$ ;

(iv) The QSDE in  $h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$

$$V_t = I_{h \otimes \Gamma} + \int_0^t V_s \sum_{j=1}^{\infty} \{ \tilde{R}_j dA_j^+(s) - \tilde{R}_j^* dA_j(s) \} + \int_0^t V_s \mathcal{L}_2 ds, \quad (13)$$

admits a unique unitary solution satisfying  $\langle u, \mathbb{E}(V_t(x \otimes I_{\Gamma}) V_t^*) v \rangle = \langle u, T_t(x) v \rangle$ , for  $u, v \in h, x \in \mathcal{A}, t \geq 0$ .

Thus in the situations where the structure of the von Neumann algebra is such that there is a group action on it consistent with a trace on it as well as with the QDS, we can construct a quantum stochastic dilation on it as was done for a QDS with bounded generator in the previous section. These hypotheses are satisfied for all classical symmetric Laplacians on Lie groups themselves, driving classical diffusions, and also for the QDS generated by the canonical non-commutative Laplacian on the Weyl  $C^*$ -algebra on  $\mathbb{R}^{2d}$  (see page 207 of ref. 6).

## Application and conclusion

We shall briefly discuss here about a physical model where QSDE with unbounded coefficients arise naturally. Consider the well-known example of a classical damped harmonic oscillator described by the equation of motion:

$$\frac{d^2 q}{dt^2} + 2\alpha \frac{dq}{dt} + \omega^2 q = 0, \quad (14)$$

where  $\alpha, \omega$  are positive constants satisfying  $\alpha < \omega$ . This is a nonconservative system, and thus cannot be described by a Hamiltonian. Nevertheless, we can introduce a pair of ‘conjugate variables’  $(p, q)$  satisfying damped harmonic oscillator described by the equations of motion:

$$\frac{dq}{dt} = p - \alpha q, \quad \frac{dp}{dt} = \delta^2 q - \alpha p, \quad (15)$$

where  $\delta = \sqrt{\omega^2 - \alpha^2}$ . It can be verified by simple computation that the second variable  $q$  of the pair  $(p, q)$  satisfying eq. (16) will be a solution of eq. (15).

Furthermore, we can rewrite eq. (16) in a more convenient form in terms of a complex-valued function  $a = a(t)$  of time  $t$ :

$$\frac{da}{dt} = -(\alpha + i\delta)a. \quad (16)$$

Here  $(p, q)$  are related to  $a$  by

$$a = (2\delta)^{-(1/2)}(p - i\delta q), \quad a^\dagger = (2\delta)^{-(1/2)}(p + i\delta q). \quad (17)$$

with  $a^\dagger(t) := \overline{a(t)}$ . Solving this equation, we get

$$p(t) = e^{-\alpha t} (p_0 \cos \delta t - \delta q_0 \sin \delta t),$$

$$q(t) = e^{-\alpha t} \left( q_0 \cos \delta t + \frac{p_0}{\delta} \sin \delta t \right),$$

where  $p_0, q_0$  are the initial values. It is clear that even if  $(p_0, q_0)$  is a true conjugate pair,  $(p(t), q(t))$  is not so for any positive  $t$ . This is of course expected. The quantization of the problem does not bring any change to the above situation and leads to the conclusion that there is no unitary time evolution which gives rise to the equation of motion (17).

Let us now change the picture and consider the situation when the damping comes from a quantum noise. To be more precise, let us replace the commutative variables  $(p, q)$  by a pair of unbounded operators, say  $(P, Q)$ , such that the operator  $a, a^*$  defined by  $a = (2\delta)^{-1/2} (P - i\delta Q)$ ,  $a^* = (2\delta)^{-1/2} (P + i\delta Q)$ , satisfying the canonical commutation relation (CCR)  $[a, a^*] = I$ . In fact, we can take  $P = -i(d/dx)$  and  $Q$  to be the multiplication by  $x$  on the Hilbert space  $h = L^2(\mathbb{R})$ . We model the quantum damped harmonic oscillator by the equation of evolution given by the following QSDE:

$$U_t = I + \int_0^t U_s (R^* A(ds) + S(A^* ds) - (\alpha + i\delta) a a^* a ds),$$

where  $R = a\sqrt{2\alpha}, S = -R$ .

It should be noted here that if  $\alpha = 0$ , that is, if there is no damping, the above equation becomes  $dU_t/dt = -i\omega U_t a^* a$ ,  $U_0 = I$ ; so that the solution will be the evolution group for the well-known standard quantum harmonic oscillator.

Some results are known on the unitarity of the solution of the above QSDE, and it has been shown on page 182 of ref. 6 that the above initial value problem does indeed admit a unitary solution.

In this useful example the technique developed earlier works very well. A part of the reason is that the potential function involved is quadratic (more generally a polynomial) in the position variable  $q$ . For a more general situation with an arbitrary smooth bounded potential, for example, one needs to adapt the theory somewhat and this will be the content of a forthcoming communication.

From an intuitive point of view, a contractive QDS  $\{T_t\}_{t \geq 0}$  is believed to describe a dissipative, irreversible evolution of an observed system. It is also believed that isolated systems, on the other hand, will undergo an automorphic evolution which is necessarily reversible. Therefore, one can infer that the irreversible evolution is likely to be due to the presence of an environment, interacting with and much larger than the observed system, in such a way that the total system (consisting of the environment and the observed system) evolves with the help of unitary conjugation in an appropriate Hilbert space de-

scribing the total system. The theory that has been developed here gives such an evolution (though stochastic) which is the description of the evolution of the total system in ‘some approximation’. But when the considerations are restricted to the observed system only, one has to ‘wash out’ or average the effect of the environment by taking expectation over the stochastic variables modelling the environment, resulting in the QDS which is the observed dissipative evolution of the observed system.

One can ask that under the total evolution, how does the stochastic variables modelling the environment behave. In fact, the fluctuation in these variables (which in general would depend on the ‘thermal state’ or temperature of the environment) and the resulting dissipation in the evolution of the observed system should be related: physicists call such result ‘fluctuation–dissipation theorems’.

Finally, just as in the study of classical stochastic process, the geometry of the underlying manifold dictates the Laplacian which in its turn drives the Brownian motion of the manifold, similar constructions can be made in some

elementary non-commutative geometric spaces<sup>6</sup>. However, there is no general theory for this at present.

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ACKNOWLEDGEMENT. This work has been partly supported by the Bhatnagar Fellowship of CSIR, New Delhi.

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