An introduction to inflation and cosmological perturbation theory*

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This article provides an introductory review of inflation and cosmological perturbation theory. I begin by motivating the need for an epoch of inflation during the early stages of the radiation dominated era, and describe how inflation is typically achieved using scalar fields. Then, after an overview of linear cosmological perturbation theory, I derive the equations governing the perturbations, and outline the generation of scalar and tensor perturbations during inflation. I illustrate that slow-roll inflation naturally leads to an almost scale-invariant spectrum of perturbations, a prediction that seems to be in good agreement with the measurements of the anisotropies of the cosmic microwave background. I describe the constraints from the recent observations on some of the more popular models of inflation. I conclude with a brief discussion on the status and certain prospects of the inflationary paradigm.

Keywords: Anisotropies, cosmological perturbation theory, cosmic microwave background, inflation.

Introduction

A major drawback of the Hot Big Bang model

THE prevailing theory about the origin and evolution of our universe is the so-called Hot Big Bang model. The model is based on two crucial observations: the discovery of the expansion of the universe as characterized by the Hubble's law, and the existence of an exceedingly isotropic and a perfectly thermal cosmic microwave background (CMB) radiation. Since the energy density of radiation falls faster with the expansion than that of matter, these two observations immediately point to the fact that the universe has expanded from a hot and dense early phase, when radiation, rather than matter, was dominant. The transition to the more recent matter-dominated epoch occurs when the radiation density falls sufficiently low such that the photons cease to interact with matter. The CMB is nothing but the relic radiation which is reaching us today from this epoch of decoupling. While fairly isotropic, the CMB possesses small anisotropies (of about one part in 10^5), and the observed pattern of the fluctuations in the CMB provides a direct snapshot of the universe at this epoch. The Hot Big Bang model has been rather successful in predicting, say, the primordial abundances of the light elements in terms of a single parameter, viz. the baryon-to-photon ratio, and the value required to fit these observations matches the value that has been arrived at independently from the structure of the anisotropies in the CMB. Despite the success of the Hot Big Bang model in explaining the results from different observations, the model has a serious drawback. Under the model, the CMB photons arriving at us today from sufficiently widely separated directions in the sky could not have interacted at the time of decoupling. Nevertheless, one finds that the temperature of the CMB photons reaching us from any two diametrically opposite directions hardly differs. (For a detailed description of the various successes and the few shortcomings of the Hot Big Bang model, and the original references for the different points mentioned above, I would refer the reader to the following texts (refs 1–8).)

The scope and success of inflation

Inflation – which refers to a period of accelerated expansion during the early stages of the radiation-dominated epoch – provides a satisfactory resolution to the above-mentioned shortcoming of the Hot Big Bang model. In fact, in addition to offering an elegant explanation for the extent of homogeneity and isotropy of the background universe, inflation also provides an attractive causal mechanism to generate the inhomogeneities superimposed upon it. The inflationary epoch amplifies the tiny quantum fluctuations present at the beginning of the epoch and converts them into classical perturbations, which leave their imprints as anisotropies in the CMB. These anisotropies in turn act as seeds for the formation of the large-scale structures that we observe at the present time as galaxies and clusters of galaxies. With the anisotropies in the CMB being measured to greater and greater precision, we have an unprecedented scope to test the predictions of inflation. As I shall discuss, the simplest models

*I will restrict my discussion to *linear* perturbation theory. Considering higher order perturbations, as is done, say, while studying non-Gaussianities, though currently attracting a lot of attention, is beyond the scope of this introductory review.

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of inflation driven by a single, slowly rolling scalar field, generically predict a nearly scale-invariant spectrum of primordial perturbations, which seems to be in good agreement with the recent observations of the CMB. (For a discussion on these different aspects of the inflationary paradigm, in addition to refs (1–8), see refs (9–23) as well.)

The organization of this review

This article presents an introductory survey of inflation and linear cosmological perturbation theory, and is organized as follows. In the next section, I shall describe the main shortcoming of the Hot Big Bang model, viz. the horizon problem. Then I shall outline how inflation helps overcome the horizon problem, and illustrate how inflation can be driven with scalar fields. I shall also introduce the concept of slow-roll inflation, and discuss the solutions to the background equations for a certain class of inflationary models in the slow-roll approximation. Next, I shall present an overview of linear perturbation theory. I shall explain the classification of the perturbations into scalars, vectors and tensors, and derive the equations governing these perturbations. I shall also discuss the behaviour of the scalar perturbations in a particular limit. After describing the generation of perturbations during inflation, I shall calculate the scalar and tensor spectra that arise in the power law and slow-roll inflation. Then, I shall discuss how such spectra compare with the recent observations, and point out the constraints on some of the more popular models of inflation. Finally, I shall conclude with a brief discussion on the status and some prospects of the inflationary paradigm.

Conventions and notations

Before I get down to brass tacks, let me list the various conventions and notations that I shall adopt. Unless I mention otherwise, I shall work in (3 + 1)-dimensions, and I shall adopt the metric signature of (+, –, –, –). While Greek indices shall denote all the spacetime coordinates, Latin indices shall refer to the spatial coordinates. I shall set $\hbar = c = 1$, but shall display G explicitly, and define the Planck mass to be $M_{\mathrm{P}} = (8\pi G)^{-1/2}$. I shall express the various quantities in terms of either the cosmic time t or the conformal time η, as is convenient. An overdot and an overprime shall denote differentiation with respect to the cosmic and conformal time coordinates of the Friedmann metric that describes the expanding universe. It is useful to note here that, for any given function, say, f, $\dot{f} = (f'/a)$ and $\ddot{f} = [(f''/a^2) - (f'a'/a^3)]$, where a is the scale factor associated with the Friedmann metric. Lastly, since observations indicate that the universe has a rather small curvature, as it is usually done in the context of inflation, I shall work with the spatially flat Friedmann model.

A few words on the references

The different texts[1–8] and the variety of reviews[9–23] that I have already mentioned discuss the motivations for the inflationary scenario, the different models of inflation, the formulation of cosmological perturbation theory, and also how the various models compare with the recent observations. In what follows, I shall typically refer to one or more of these texts or reviews. I have also tried to refer to the original papers. However, I should stress that these references are often representative and not necessarily exhaustive.

The horizon problem

Let me begin by describing the horizon problem in the Hot Big Bang model. Consider a spatially flat, smooth, Friedmann universe described by line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2 = a^2(\eta)\,(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2), \tag{1}$$

where t is the cosmic time, $a(t)$ the scale factor, and $\eta = \int [\mathrm{d}t/a(t)]$ denotes the conformal time coordinate. In such a background, the horizon, viz. the size of a causally connected region, is defined as the physical radial distance travelled by a light ray from the Big Bang singularity at $t = 0$ up to a given time t. The horizon can be expressed in terms of the scale factor $a(t)$ as follows[6,7]:

$$h(t) = a(t)\int_0^t \frac{\mathrm{d}\tilde{t}}{a(\tilde{t})}. \tag{2}$$

Let me now compare the linear dimension of the forward and the backward light cones at the time of decoupling in the Hot Big Bang model.

If one assumes that the universe was dominated by non-relativistic matter from the time of decoupling t_{dec} until today t_0, then the physical size of the region on the last scattering surface from which we receive the CMB is given by

$$\ell_{\mathrm{B}}(t_0, t_{\mathrm{dec}}) = a_{\mathrm{dec}} \int_{t_{\mathrm{dec}}}^{t_0} \frac{\mathrm{d}\tilde{t}}{a(\tilde{t})} \simeq 3(t_{\mathrm{dec}}^2 t_0)^{1/3}, \tag{3}$$

where a_{dec} denotes the value of the scale factor at decoupling, and I have used the fact that $t_0 \gg t_{\mathrm{dec}}$ in arriving at the final expression. On the other hand, if I assume that the universe was radiation dominated from the Big Bang until the epoch of decoupling, then the linear dimension of the horizon at decoupling turns out to be

$$\ell_{\mathrm{F}}(t_{\mathrm{dec}}, 0) = a_{\mathrm{dec}} \int_0^{t_{\mathrm{dec}}} \frac{\mathrm{d}\tilde{t}}{a(\tilde{t})} = (2t_{\mathrm{dec}}). \tag{4}$$

The ratio, say, R, of the linear dimensions of the backward and the forward light cones at decoupling then reduces to[20]

$$R \equiv \left(\frac{\ell_{\mathrm{B}}}{\ell_{\mathrm{F}}}\right) = \left(\frac{3}{2}\right)\left(\frac{t_0}{t_{\mathrm{dec}}}\right)^{1/3} \simeq 70, \tag{5}$$

and the last equality follows from the observational fact that $t_0 \simeq 10^{10}$ years, while $t_{\mathrm{dec}} \simeq 10^5$ years. In other words, the linear dimension of the backward light cone is about 70 times larger than the forward light cone at decoupling. Despite this, the CMB turns out to be extraordinarily isotropic. This is the horizon problem.

There is another way of stating the horizon problem. Note that the physical wavelengths associated with the perturbations, say, λ_{P}, always grow as the scale factor, i.e. $\lambda_{\mathrm{P}} \propto a$. In contrast, in a power law expansion of the form $a(t) \propto t^q$, the Hubble radius, viz. $d_{\mathrm{H}} = H^{-1} = (\dot{a}/a)^{-1}$, goes as $a^{1/q}$, so that we have $(\lambda_{\mathrm{P}}/d_{\mathrm{H}}) \propto a^{[(q-1)/q]}$. [I consider the Hubble radius rather than the horizon size since, in any power law expansion, the Hubble radius is equivalent to the horizon up to a finite multiplicative constant. Also, being a local quantity, the Hubble radius often proves to be more convenient to handle than the horizon.] This implies that, when $q < 1$ – a condition which applies to both the radiation and the matter dominated epochs – the physical wavelength grows faster than the corresponding Hubble radius as *we go back in time*[1]. In other words, the primordial perturbations need to be correlated on scales much larger than the Hubble radius at sufficiently early times in order to result in the anisotropies that we observe in the CMB. Consequently, within the Hot Big Bang model, any mechanism that is invoked to generate the primordial fluctuations has to be intrinsically acausal.

The inflationary scenario

Actually, apart from the horizon problem, there also exist a few other puzzles to which the Hot Big Bang model is unable to provide a satisfactory solution. These include, the extent to which the universe happens to be spatially flat today (to about one part in 10^2), and the unacceptable density of relics that would have been formed when high-energy symmetries were broken in the early universe, to name just two. Amongst all these issues, the horizon problem is arguably the most significant. Moreover, it so happens that the inflationary solution to the horizon problem also aids in surmounting the other difficulties as well[24–29]. For these reasons, I shall restrict my attention to the horizon problem. In the following sub-sections, after illustrating how inflation helps in overcoming the horizon problem, I shall outline how inflation is typically achieved with scalar fields.

Bringing the modes inside the Hubble radius

As I discussed above, length scales of cosmological interest today (say, $1 \lesssim \lambda_0 \lesssim 10^4$ Mpc), enter the Hubble radius either during the radiation or the matter dominated epochs, and are outside the Hubble radius at earlier times. If a causal mechanism is to be responsible for the origin of the inhomogeneities, then, clearly, these length scales should be inside the Hubble scales (i.e. $\lambda_{\mathrm{P}} < d_{\mathrm{H}}$) in the very early stages of the universe. This will be possible provided we have an epoch in the early universe during which λ_{P} decreases faster than the Hubble radius *as we go back in time*, i.e. if we have[5,23]

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda_{\mathrm{P}}}{d_{\mathrm{H}}}\right) < 0, \tag{6}$$

which then leads to the condition that

$$\ddot{a} > 0. \tag{7}$$

In other words, the universe needs to undergo a phase of accelerated (inflationary) expansion during the early stages of the radiation-dominated epoch, if a physical mechanism is to account for the generation of the primordial fluctuations.

In order to illustrate these points, in Figure 1, I have plotted log (length), where 'length' denotes either the physical wavelength of a mode or the Hubble radius, against log a, during inflation and the radiation dominated epochs[1]. For convenience, I have chosen to describe inflation by the power law expansion $a \propto t^q$, with $q > 1$. In such a situation, while $\lambda_{\mathrm{P}} \propto a$ during all the epochs, the

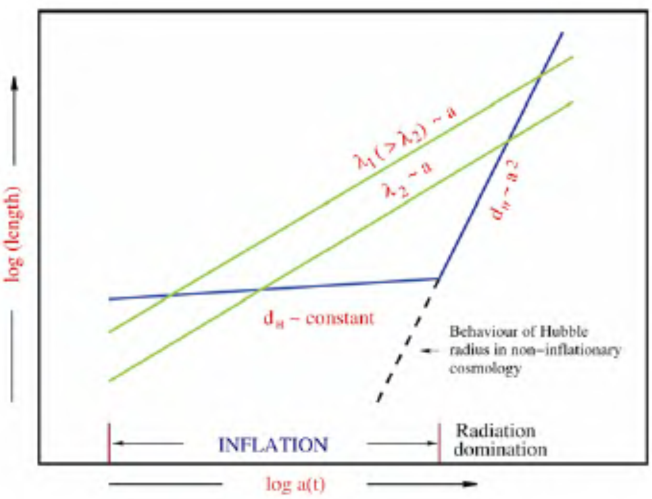

Figure 1. Evolution of physical wavelength λ_{P} (in green) and the Hubble radius d_{H} (in blue) plotted as a function of the scale factor a on a logarithmic plot during the inflationary and radiation-dominated epochs. As discussed in the text, in such a plot, the slope of the Hubble length is much less than unity during inflation, a feature which allows to bring around the modes inside the Hubble radius at a suitably early time.

Hubble length d_H behaves as $a^{1/q}$ and a^2 during inflation and radiation-dominated epochs. Evidently, all these quantities will be described by straight lines in the above plot. Whereas the straight lines describing the evolution of the physical wavelengths will have a unit slope, those describing the Hubble radius during the inflationary and radiation-dominated epochs will have a slope of much less than unity (say, for $q >> 1$) and two respectively. It is then clear from Figure 1 that, *as we go back in time*, modes that leave the Hubble radius during the later epochs will not be inside the Hubble radius during the early stages of the universe *unless we have a period of inflation*.

How much inflation do we need?

Let us now try and understand as to how much inflation we shall require in order to ensure that the forward lightcone from the Big Bang to the epoch of decoupling is at least as large as the backward light cone from today to the epoch of decoupling, thereby resolving the horizon problem. For simplicity, let me assume that the universe undergoes exponential inflation, say, from time t_i to t_f, during the early stages of the radiation-dominated epoch. Let H be the constant Hubble scale during exponential inflation, and A denote the extent by which the scale factor increases during the inflationary epoch. For $A >> 1$, the dominant contribution to the size of the horizon at decoupling arises due to the rapid expansion during inflation, and it can be evaluated to be

$$\ell_{\mathrm{I}}(t_{\mathrm{dec}}, 0) = a_{\mathrm{dec}} \int_0^{t_{\mathrm{dec}}} \frac{\mathrm{d}\tilde{t}}{a(\tilde{t})} \simeq \left(\frac{a_{\mathrm{dec}}}{H}\right)\left(\frac{t_{\mathrm{dec}}}{t_f}\right)^{1/2} A, \tag{8}$$

where I have set $t_i \simeq H^{-1}$. In such a case, the ratio of the forward and the backward lightcones at the epoch of decoupling is given by[4]

$$R_{\mathrm{I}} \equiv \left(\frac{\ell_{\mathrm{I}}}{\ell_{\mathrm{B}}}\right) \simeq \left(\frac{A}{10^{26}}\right), \tag{9}$$

and, in arriving at the final number, I have chosen $H \simeq 10^{13}$ GeV. Clearly, $R_{\mathrm{I}} \simeq 1$, if $A \simeq 10^{26}$. Given the scale factor, the amount of expansion that has occurred from an initial time t_i to a time t is usually expressed in terms of the number of e-folds defined as follows:

$$N = \int_{t_i}^{t} \mathrm{d}t\, H = \ln\left(\frac{a(t)}{a_i}\right). \tag{10}$$

Since $\ln A = \ln(a_f/a_i) \simeq 60$, it is often said that one requires about 60 e-folds of inflation to overcome the horizon problem. [In fact, 60 e-folds is roughly an observational upper bound which ensures that the largest scale today is inside the Hubble radius during the inflationary epoch[30,31]. Actually, the number of e-folds needed to resolve the horizon problem depends on the energy scale at which inflation takes place. For instance, if inflation is assumed to occur at a rather low energy scale of, say, $H \simeq 10^2$ GeV, then, even 50 e-folds will suffice to surmount the horizon problem.]

Driving inflation with scalar fields

If ρ and p denote the energy density and pressure of the smooth component of the matter field that is driving the expansion, then the Einstein's equations corresponding to the line element in eq. (1) result in the following two Friedmann equations for the scale factor $a(t)$:

$$H^2 = \left(\frac{8\pi G}{3}\right)\rho, \tag{11}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\left(\frac{4\pi G}{3}\right)(\rho + 3p), \tag{12}$$

where, as I had mentioned, $H = (\dot{a}/a)$ is the Hubble parameter. It is clear from the second Friedmann equation that, for $\ddot{a}$ to be positive, we require that $(\rho + 3p) < 0$. Neither ordinary matter (corresponding to $p = 0$) nor radiation (which corresponds to $p = (\rho/3)$) satisfies this condition. In such a situation, we need to identify another form of matter to drive inflation.

Scalar fields, which are often encountered in various models of high-energy physics, can easily help achieve the necessary condition, thereby leading to inflation. Consider a canonical scalar field, say, ϕ, described by the potential $V(\phi)$. Such a scalar field is governed by the action

$$S[\phi] = \int \mathrm{d}^4x \sqrt{-g} \left[\left(\frac{1}{2}\right)(\partial_\lambda \phi \partial^\lambda \phi) - V(\phi)\right], \tag{13}$$

with the associated stress-energy tensor being given by

$$T^{\mu}_{\nu} = \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \left[\left(\frac{1}{2}\right)(\partial_\lambda \phi \partial^\lambda \phi) - V(\phi)\right]. \tag{14}$$

The symmetries of the Friedmann background, viz. homogeneity and isotropy, imply that the scalar field will depend only on time and, hence, the resulting stress energy tensor will be diagonal. Therefore, the energy density ρ and the pressure p associated with the scalar field simplify to

$$T^0_0 = \rho = \left[\left(\frac{\dot{\phi}^2}{2}\right) + V(\phi)\right], \tag{15}$$

$$T^i_j = -p\delta^i_j = -\left[\left(\frac{\dot{\phi}^2}{2}\right) - V(\phi)\right]\delta^i_j. \quad (16)$$

Moreover, from eq. (13), one can arrive at the following equation of motion for the scalar field ϕ in the Friedmann universe:

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0. \quad (17)$$

where $V_\phi = (\mathrm{d}V/\mathrm{d}\phi)$. From the above expressions for ρ and p, one finds that the condition for inflation, viz. $(\rho + 3p) < 0$, reduces to

$$\dot{\phi}^2 < V(\phi). \quad (18)$$

In other words, inflation can be achieved if the potential energy of the scalar field dominates its kinetic energy.

Given a $V(\phi)$ that is motivated by a high-energy model, the first Friedmann equation (eq. (11)) and eq. (17) that governs the evolution of the scalar field have to be consistently solved for scale factor and the scalar field, with suitable initial conditions. However, using eqs (15) and (16) for the energy density and pressure associated with the scalar field, the Friedmann equations (11) and (12) can be rewritten as

$$H^2 = \left(\frac{1}{3M_P^2}\right)\left[\left(\frac{\dot{\phi}^2}{2}\right) + V(\phi)\right], \quad (19)$$

$$\dot{H} = -\left(\frac{1}{2M_P^2}\right)\dot{\phi}^2, \quad (20)$$

where, for convenience, I have set $(8\pi G) = M_P^{-2}$, as defined earlier. These two equations can then be combined to express the scalar field and potential parametrically in terms of the cosmic time t as follows[4]:

$$\phi(t) = \sqrt{2}M_P\int \mathrm{d}t\sqrt{(-\dot{H})}, \quad (21)$$

$$V(t) = M_P^2(3H^2 + \dot{H}). \quad (22)$$

If we know the scale factor $a(t)$, these two equations allow us to 'reverse engineer' the potential from which such a scale factor can arise. Using this procedure, I shall now 'reconstruct' the potentials of two commonly considered models of inflation.

Consider the power law expansion

$$a(t) = a_1 t^q, \quad (23)$$

with $q > 1$ corresponding to inflation, a_1 being an arbitrary constant. On substituting this scale factor in eq. (21) and, upon integration, one can immediately show that the scalar field evolves as

$$\left(\frac{\phi(t)}{M_P}\right) = \sqrt{(2q)}\ln\left[\sqrt{\left(\frac{V_0}{(3q-1)q}\right)}\left(\frac{t}{M_P}\right)\right], \quad (24)$$

where V_0 is a constant of integration. The potential that leads to such a behaviour can then be obtained using eq. (22), and the above expressions for $a(t)$ and $\phi(t)$. It is found to be[32]:

$$V(\phi) = V_0 \exp\left[-\sqrt{\frac{2}{q}}\left(\frac{\phi}{M_P}\right)\right]. \quad (25)$$

In a similar fashion, it is straightforward to establish that the potential

$$V(\phi) = (3\alpha^2\beta^2\Gamma^\kappa M_P^2)\left[1 - \left(\frac{\kappa^2}{6}\right)\left(\frac{M_P}{\phi}\right)^2\right]\left(\frac{\phi}{M_P}\right)^{-\kappa}, \quad (26)$$

where $\Gamma = \sqrt{(2\alpha\kappa)}$ and $\kappa = [4(1-\beta)/\beta]$, leads to the following behaviour for $a(t)$:

$$a(t) = a_1 \exp(\alpha t^\beta), \quad (27)$$

with $\alpha > 0$ and $0 < \beta < 1$, and a_1 is again some arbitrary constant. Since this scale factor grows faster than the power law inflation, but slower than exponential expansion, it is referred to as intermediate inflation[33,34].

Slow roll inflation

The condition given in eq. (18), viz. that the potential energy of the inflaton dominates the kinetic energy, is necessary for inflation to take place. [It is common to refer to the scalar field that drives inflation as the 'inflaton'.] However, inflation is *guaranteed*, if the field rolls slowly down the potential such that

$$\dot{\phi}^2 \ll V(\phi). \quad (28)$$

Moreover, it can be ensured that the field is slowly rolling for a sufficiently long time (to achieve the required 60 or so e-folds of inflation), provided

$$\ddot{\phi} \ll (3H\dot{\phi}). \quad (29)$$

These two conditions lead to the slow-roll approximation[35–37], which, as we shall see, allows one to construct analytical solutions, both for the background and the perturbations. The approximation is usually described in terms of what are referred to as the slow-roll parameters. Two types of slow-roll parameters – the potential slow-roll parameters and the Hubble slow-roll parameters – are often considered in the literature [Though, I should add that, nowadays, it seems to be more common to use the following hierarchy of Hubble flow functions[38,39]:

$$\epsilon_0 \equiv \left(\frac{H_*}{H}\right) \quad \text{and} \quad \epsilon_{i+1} \equiv \left(\frac{\mathrm{d}\ln|\epsilon_i|}{\mathrm{d}N}\right) \quad \text{for } i \geq 0,$$

with H_* being the Hubble parameter evaluated at some given time during inflation.] I shall describe these two sets of parameters below, and discuss the solutions for the inflaton and the scale factor for a particular class of potentials in the slow-roll approximation.

The potential slow-roll parameters: When provided with a potential $V(\phi)$, the potential slow-roll approximation corresponds to requiring the following two dimensionless parameters[35–37]:

$$\epsilon_V = \left(\frac{M_P^2}{2}\right)\left(\frac{V_\phi}{V}\right)^2 \quad \text{and} \quad \eta_V = M_P^2\left(\frac{V_{\phi\phi}}{V}\right), \tag{30}$$

where $V_{\phi\phi} \equiv (\mathrm{d}^2V/\mathrm{d}\phi^2)$, to be small when compared to unity. The two quantities ϵ_V and η_V are referred to as the potential slow-roll (PSR) parameters. It is straightforward to show that neglecting the kinetic energy term $(\dot{\phi}^2/2)$ in the Friedmann eq. (19) and the acceleration term $\ddot{\phi}$ in the equation of motion (eq. (17)) for the scalar field is equivalent to the smallness of these parameters. However, it should be emphasized that the converse is not true. The smallness of the PSR parameters is only a necessary condition, and it is not sufficient to ensure that these terms can indeed be ignored. The reason being that the PSR parameters only restrict the form of the potential, and not the dynamics of the solutions. Even if ϵ_V and η_V are small, there is no assurance that inflation will take place since the value of $\dot{\phi}$ can be as large as possible. Therefore, in addition to the two PSR parameters being small, the slow-roll approximation actually requires the additional condition that the scalar field is moving slowly along the attractor solution determined by the equation: $(3H\dot{\phi}) = -V_\phi$ (for a detailed discussion on this point, see Liddle *et al.*[40]).

In spite of this shortcoming, the PSR parameters often prove to be handy. For instance, given a potential, they immediately allow us to determine the domains and parameters of the potential that can lead to inflation. I shall now discuss two examples to illustrate the utility of these parameters. Consider potentials of the form[41]:

$$V(\phi) = (V_0\phi^n), \tag{31}$$

where V_0 is a constant and $n > 0$. Let us restrict ourselves to the region $\phi > 0$, wherein $V(\phi)$ is positive for all n. It is straightforward to show that the slow-roll conditions (viz. $(\epsilon_V, \eta_V) \ll 1$) are satisfied when ϕ is much greater than M_P. Since inflation occurs for such large values of the field, these potentials are often classified as 'large field' models[20]. Now, consider the following potential which describes the pseudo-Nambu–Goldstone boson

$$V(\phi) = \Lambda[1 + \cos(\phi/f)], \tag{32}$$

where Λ and f are constants that characterize the depth and the width of the potential. This potential 'naturally' leads to inflation for values of the field that are small when compared to the Planck scale[42]. Hence, such models are usually referred to as 'small field' models.

The Hubble slow-roll parameters: The Hubble slow-roll (HSR) parameters turn out to be a better choice to describe the slow-roll approximation than the PSR parameters, since they do not require any additional conditions to be satisfied for the approximation to be valid. The HSR parameters are called so, since they are defined in terms of the Hubble parameter H, which is treated as a function of the scalar field ϕ (ref. 36). In such a case, we can write eq. (20) as

$$\dot{\phi} = -(2M_P^2)H_\phi, \tag{33}$$

where $H_\phi = (\mathrm{d}H/\mathrm{d}\phi)$. This expression can then be used to rewrite the first Friedmann equation (eq. (19)) as follows:

$$H_\phi^2 - \left(\frac{3H^2}{2M_P^2}\right) = -\left(\frac{V}{2M_P^4}\right), \tag{34}$$

a relation that is referred to as the Hamilton–Jacobi formulation of inflation[36].

Taking $H(\phi)$ to be the primary quantity, the dimensionless HSR parameters ϵ_H and δ_H are defined as follows[40]:

$$\epsilon_H = (2M_P^2)\left(\frac{H_\phi}{H}\right)^2 \quad \text{and} \quad \delta_H = (2M_P^2)\left(\frac{H_{\phi\phi}}{H}\right), \tag{35}$$

where $H_{\phi\phi} \equiv (\mathrm{d}^2H/\mathrm{d}\phi^2)$. Using eqs (17), (33) and (34), these two parameters can be written as

$$\epsilon_H = \left(\frac{3\dot{\phi}^2}{2\rho}\right) = -\left(\frac{\dot{H}}{H^2}\right), \tag{36}$$

$$\delta_H = -\left(\frac{\ddot{\phi}}{H\dot{\phi}}\right) = \epsilon_H - \left(\frac{\dot{\epsilon}_H}{2H\epsilon_H}\right), \tag{37}$$

where ρ is the energy density associated with the scalar field. The following points are clear from these expressions. Firstly, $\epsilon_H \ll 1$ is precisely the condition required for neglecting the kinetic energy term in the total energy of the scalar field. Secondly, the limit $\delta_H \ll 1$ corresponds to the situation wherein the acceleration term of the scalar field can be ignored in eq. (17) when compared to the term involving the velocity. Finally, the inflationary condition $\ddot{a} > 0$ *exactly* corresponds to $\epsilon_H < 1$.

It should again be emphasized that, since the smallness of the HSR parameters ensure that $\dot{\phi}$ is small, the HSR approximation implies the PSR approximation. But, the converse does not hold without assuming the constraint that the inflation is already on the attractor[40].

Solutions in the slow-roll approximation: Note that the equation of motion of the scalar field (eq. (17)) and the first Friedmann equation (eq. (19)) can be written in terms of the two HSR parameters as

$$H^2\left[1-\left(\frac{\varepsilon_{\rm H}}{3}\right)\right]=\left(\frac{V}{3M_{\rm P}^2}\right), \tag{38}$$

$$(3H\dot{\phi})\left[1-\left(\frac{\delta_{\rm H}}{3}\right)\right]=-V_{\phi}. \tag{39}$$

The slow-roll approximation corresponds to the situation wherein the HSR parameters $\varepsilon_{\rm H}$ and $\delta_{\rm H}$ satisfy the following conditions:

$$\varepsilon_{\rm H} \ll 1,\ \delta_H \ll 1 \ \text{and}\ \mathcal{O}[\varepsilon_{\rm H}^2, \delta_{\rm H}^2, (\varepsilon_{\rm H}\delta_{\rm H})] \ll \varepsilon_{\rm H}. \tag{40}$$

At the leading order in the slow-roll approximation, eqs (38) and (39) above reduce to

$$H^2 \simeq \left(\frac{V}{3M_{\rm P}^2}\right) \quad \text{and} \quad (3H\dot{\phi}) \simeq -V_{\phi}. \tag{41}$$

Being first-order differential equations, given a potential, these equations can be easily integrated to obtain the solutions to the scale factor and the scalar field in the slow-roll limit. Let me now discuss the solutions in this limit to the large field models (eq. (31)) that I had considered earlier.

For the potential given in eq. (31), when $n \neq 4$, in the slow-roll limit, the solution to the scalar field is given by[8,18]

$$\phi^{[(4-n)/2]}(t) \simeq \phi_i^{[(4-n)/2]} + \sqrt{\frac{V_0}{3}}\left[\frac{n(n-4)}{2}\right]M_{\rm P}\,(t-t_i), \tag{42}$$

whereas when $n = 4$, one finds that

$$\phi(t) \simeq \phi_i \exp -[\sqrt{(V_0/3}(4M_{\rm P})(t-t_i)], \tag{43}$$

and, in both these solutions, ϕ_i is a constant that denotes the value of the scalar field at some initial time t_i. For all n, the scale factor can be expressed in terms of these solutions for the scalar field as follows:

$$a(t) \simeq a_i \exp -\left[\left(\frac{1}{2nM_{\rm P}^2}\right)(\phi^2(t)-\phi_i^2)\right], \tag{44}$$

with a_i being the value of the scalar factor at t_i. It is also useful to note that, in the slow-roll limit, eq. (41) allows us to express the number of e-folds from t_i to t during inflation as

$$N = \ln\left(\frac{a}{a_i}\right) = \int_{t_i}^{t} {\rm d}t\, H \simeq -\left(\frac{1}{M_{\rm P}^2}\right)\int_{\phi_i}^{\phi} {\rm d}\phi\left(\frac{V}{V_{\phi}}\right), \tag{45}$$

where the upper limit ϕ is the value of the scalar field at the time t. In terms of e-folds, for the large field models, the scalar field and the Hubble parameter are given by

$$\phi^2(N) \simeq [\phi_i^2 - (2M_{\rm P}^2 n)N], \tag{46}$$

$$H^2(N) \simeq \left(\frac{V_0 M_{\rm P}^{(n-2)}}{3}\right)\left[\left(\frac{\phi_i}{M_{\rm P}}\right)^2 - (2nN)\right]^{(n/2)}. \tag{47}$$

Essential linear, cosmological perturbation theory

Though the inflationary scenario was originally proposed to resolve the different puzzles related to the smooth background, it was soon realized that it also offers a simple mechanism to generate the primordial perturbations. Before I turn to demonstrating how inflation produces these inhomogeneities, I shall provide an overview of essential cosmological perturbation theory. (As I had mentioned, I will restrict myself to linear perturbations.) In what follows, after a discussion on the classification of the perturbations into scalars, vectors and tensors, I shall derive the equations governing these perturbations. Since the scalar perturbations are primarily responsible for the anisotropies in the CMB and the formation of structures, I shall also briefly highlight the evolution of these perturbations at super-Hubble scales during the radiation and matter-dominated epochs.

Classification of the perturbations

CMB observations indicate that the anisotropies at the epoch of decoupling are rather small (one part in 10^5, as I had mentioned). If so, the amplitude of the deviations from homogeneity will be even smaller at earlier epochs. This suggests that the generation and evolution of the perturbations (until structures begin to form late in the matter-dominated epoch), can be studied using linear perturbation theory.

In a Friedmann background, the metric perturbations can be decomposed according to their behaviour under local rotation of the spatial coordinates on hypersurfaces of constant time. This property leads to the classification of the perturbations as scalars, vectors and tensors[43,44]. Scalar perturbations remain invariant under rotations (and, hence, can be said to have zero spin). As we shall see, these are the principal perturbations that are responsible for the anisotropies and inhomogeneities in the universe. Vector and tensor perturbations – as their names indicate – transform as vectors and tensors do under rotations (and, as a result, have spins of unity and two respectively). The vector perturbations are generated by rotational velocity fields and, therefore, are also referred to as the vorticity modes. Finally, the tensor perturbations describe gravitational waves, and it is important to note that they can exist even in the absence of sources[45,46].

The number of independent scalar, vector and tensor degrees of freedom: Let us now carry out the exercise of counting the number of degrees of freedom associated with these different types of perturbations[15]. To highlight the counting procedure, in this sub-section, I shall, in fact, work in arbitrary spacetime dimensions. In $(D + 1)$-dimensions, the metric tensor, being symmetric, has $[(D + 1)(D + 2)/2]$ degrees of freedom. However, not all of these degrees are independent since there exist $(D + 1)$ degrees of freedom associated with the coordinate transformations. If we eliminate these degrees, we are left with a total of $[D(D + 1)/2]$ independent degrees of freedom describing the metric perturbations. These degrees of freedom contain the scalar, vector and tensor perturbations.

Let $\delta g_{\mu\nu}$ denote the metric perturbations in the Friedmann universe. The perturbed metric can be split as follows:

$$\delta g_{\mu\nu} = (\delta g_{00},\ \delta g_{0i},\ \delta g_{ij}). \tag{41}$$

Since it contains no running index, evidently, the perturbation $\delta g_{00} = \mathcal{A}$, say, is a scalar. As is well known, one can always decompose a vector into a gradient of a scalar and a vector that is divergence-free. Hence, we can write δg_{0i} as

$$\delta g_{0i} = (\nabla_i \mathcal{B} + \mathcal{S}_i), \tag{49}$$

where $\mathcal{B}$ is a scalar, while $\mathcal{S}_i$ is a vector that is divergence-free, i.e. $(\nabla_i \mathcal{S}^i) = 0$. A similar decomposition can be carried for the quantity δg_{ij} by essentially repeating the above analysis on each of the two indices. Therefore, the quantity δg_{ij} can be decomposed as

$$\delta g_{ij} = \psi \delta_{ij} + (\nabla_i \mathcal{F}_j + \nabla_j \mathcal{F}_i),$$
$$+\left[\left(\frac{1}{2}(\nabla_i \nabla_j + \nabla_j \nabla_i) - \left(\frac{1}{3}\right)\delta_{ij}\nabla^2\right)\mathcal{E}\right] + \mathcal{H}_{ij}, \tag{50}$$

where ψ and $\mathcal{E}$ are scalar functions, $\mathcal{F}_i$ – like $\mathcal{S}_i$ above – is a divergence-free vector, and $\mathcal{H}_{ij}$ is a symmetric, traceless and transverse tensor that satisfies the conditions $\mathcal{H}_i^i = 0$ and $(\nabla_i \mathcal{H}^{ij}) = 0$. Let me again count the degrees of freedom of the perturbed metric tensor through the various functions that I have introduced. To describe $\delta g_{\mu\nu}$, we require the scalars $\mathcal{A}$, $\mathcal{B}$, ψ and $\mathcal{E}$, amounting to four degrees of freedom. We also need the two divergence-free spatial vectors $\mathcal{S}_i$ and $\mathcal{F}_i$ that add up to $[2(D - 1)]$ degrees. Moreover, after we impose the traceless and transverse conditions, the tensor $\mathcal{H}_{ij}$ has

$$\left[\frac{D(D+1)}{2}\right] - (D+1) = \left[\frac{(D+1)(D-2)}{2}\right], \tag{51}$$

degrees of freedom. Upon adding all these scalar, vector and tensor degrees, I obtain

$$4 + 2(D-1) + \left[\frac{(D+1)(D-2)}{2}\right] = \left[\frac{(D+1)(D+2)}{2}\right], \tag{52}$$

which, as I discussed above, are the total number of degrees of freedom associated with the perturbed metric in $(D + 1)$-dimensions.

In the same fashion, let me decompose the degrees of freedom associated with the coordinate transformations. In a Friedmann background, the $(D + 1)$ coordinate transformations that relate different coordinate systems which describe the perturbed metric can be expressed in terms of scalars, say, δt and δx, as follows:

$$t \to (t + \delta t) \quad \text{and} \quad x^i \to [x^i + \nabla^i(\delta x)]. \tag{53}$$

Similarly, one can construct coordinate transformations in terms of a divergence-free vector, say, δx^i, as

$$t \to t \quad \text{and} \quad x^i \to (x^i + \delta x^i). \tag{54}$$

(It should be emphasized that these coordinate transformations are of a particular form, and are not completely arbitrary. The form of these transformations is dictated by the fact that the difference between the coordinates is determined by the amplitude of the perturbations. In other words, it is a gauge transformation. I shall somewhat elaborate on this point below.) There is no further coordinate degrees of freedom associated with the tensor perturbations. The two scalar quantities δt and δx, and the divergence free vector δx^i, constitute the $[2 + (D - 1)] = (D + 1)$ degrees of freedom associated with the coordinate transformations.

Let me now subtract the coordinate degrees of freedom from the total degrees to arrive at the independent number of scalar, vector and tensor degrees of freedom. Four scalar functions, viz. $(\mathcal{A}, \mathcal{B}, \psi, \mathcal{E})$, were required to describe the perturbed metric tensor $\delta g_{\mu\nu}$. But, there also exist two scalar degrees of freedom associated with one's choice of coordinates, i.e. δt and δx. Hence, the actual number of independent scalar degrees of freedom is

$(4-2) = 2$. And, it is useful to notice that this is true in arbitrary spacetime dimensions. I needed two divergence-free vector functions, $\mathcal{S}_i$ and $\mathcal{F}_i$, amounting to a total of $[2(D-1)]$ degrees of freedom, to describe the perturbed metric tensor. If we subtract the $(D-1)$ vector degrees of freedom corresponding to the coordinate transformations that can be achieved through δx^i from this number, one is left with $[2(D-1)] - (D-1) = (D-1)$ true vector degrees of freedom. As I have already pointed out, the tensor perturbations contain $[(D+1)\,(D-2)/2]$ independent degrees of freedom. Upon adding these, I obtain that

$$2+(D-1)+\left[\frac{(D+1)(D-2)}{2}\right]=\left[\frac{D(D+1)}{2}\right], \tag{55}$$

which is the total number of independent degrees of freedom describing the perturbed metric tensor.

Note that, in the $(3+1)$-dimensional case of our interest, there exists two each of the scalar, vector and tensor degrees of freedom.

The decomposition theorem: I have focused above on decomposing the perturbed metric tensor into different types of perturbations. Let me now discuss the equivalent classification of the source of the metric perturbations, viz. the stress–energy tensor. Since the stress–energy tensor is a symmetric two tensor just as the metric tensor is, the perturbed stress energy tensor can also be classified as the scalar, vector and tensor components. For instance, while the perturbed inflaton and perfect fluids – such as radiation and matter – are scalar sources, velocity fields with vorticity are, evidently, vector sources. Anisotropic stresses, when the possible scalar and vector contributions have been eliminated, constitute a tensor source.

Given the perturbed metric tensor $\delta g_{\mu\nu}$, the corresponding Einstein tensor, say, $\delta G_{\mu\nu}$, can immediately be calculated at the same order in the perturbations. The Einstein equations that relate the perturbed Einstein tensor to the perturbed stress–energy tensor, say, $\delta T_{\mu\nu}$, will then lead to the equations governing the perturbations. According to the decomposition theorem[7,9], which I shall use without proof, at the linear order in the perturbations, the scalar, vector and tensor perturbations decouple and, hence, they can be analysed separately. In other words, each type of metric perturbation is affected by only the same type of source. Therefore, the three types of perturbation can be studied independent of each other.

Gauges: The Friedmann line element (eq. (1)) describes the expanding universe in the frame of a comoving observer. The comoving observer is special since, it is with respect to such an observer that the universe appears homogeneous and isotropic. However, there is no such uniquely preferred frame of reference in the presence of perturbations. A variety of coordinate choices are possible, with the only requirement being that the metric and the coordinates reduce to the standard Friedmann line element in the limit when the perturbations vanish. As a result, the difference between the various coordinates have the same amplitude as the perturbations themselves. A particular choice of coordinates is called a gauge. If one changes the coordinate system, one would obtain another metric, i.e. a different gauge. The transformation from one gauge to another (such as eqs (53) and (54)) is referred to as a gauge transformation. There exist two approaches to studying the evolution of the perturbations. Either one constructs gauge-invariant quantities, or chooses a particular gauge and works in the specific gauge throughout. Being simpler and more wieldy, I shall adopt the latter approach.

Scalar perturbations

I shall now turn to deriving the equations of motion governing the three different types of perturbation. Let me first consider the case of the scalar perturbations.

The longitudinal gauge and the perturbed Einstein tensor: As mentioned above, I shall work in a particular gauge to describe the various perturbations. A convenient gauge to describe the scalar perturbations is the longitudinal gauge. If we take into account the scalar perturbations to the background metric, i.e. eq. (1), then in this gauge, the Friedmann line element is given by

$$\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - a^2(t)(1-2\Psi)\mathrm{d}\mathbf{x}^2, \tag{56}$$

where Φ and Ψ are the two independent functions that describe the perturbations. (Note that this choice of gauge corresponds to $\mathcal{A} \propto \Phi$, $\psi \propto \Psi$ and $\mathcal{B} = \mathcal{E} = 0$.) At the linear order in the perturbations, the components of the perturbed Einstein tensor, viz. δG^μ_ν, corresponding to the line element, i.e. eq. (56) above can be evaluated to be[20]

$$\delta G^0_0 = -6H(\dot{\Psi}+H\Phi)+\left(\frac{2}{a^2}\right)\nabla^2\Psi, \tag{57}$$

$$\delta G^0_i = 2\nabla_i(\dot{\Psi}+H\Phi), \tag{58}$$

$$\delta G^i_j = -2\left[\ddot{\Psi}+H(3\dot{\Psi}+\dot{\Phi})+(2\dot{H}+3H^2)\Phi + \left(\frac{1}{a^2}\right)\nabla^2\mathcal{D}\right]\delta^i_j+\left(\frac{1}{a^2}\right)\nabla^i\nabla_j\mathcal{D}. \tag{59}$$

where $\mathcal{D} = (\Phi - \Psi)$.

Equations of motion: The only sources of perturbations that I shall consider in this review will be scalar fields and perfect fluids. As I shall illustrate in the next section, scalar fields do not possess any anisotropic stress at the linear order in the perturbations. I shall assume that the perfect fluids that I consider do not contain any anisotropic stresses either. Under these conditions, the perturbed stress-energy tensor associated with such sources can be expressed as follows:

$$\delta T_0^0 = \delta\rho,\ \ \delta T_i^0 = (\nabla_i \delta\sigma)\ \ \text{and}\ \ \delta T_j^i = -\delta p\, \delta_j^i, \tag{60}$$

where $\delta\rho$, $\delta\sigma$ and δp are the scalar quantities that denote the perturbations in the energy density, momentum flux, and pressure respectively. The first-order Einstein's equations, viz. $\delta G_\nu^\mu = (8\pi G)\delta T_\nu^\mu$, then lead to the required set of equations governing the perturbations. It is clear from the form of δG_j^i above that, in the absence of anisotropic stresses, the corresponding Einstein equation leads to: $\Phi = \Psi$. In such a case, the remaining three Einstein equations simplify to[20]:

$$-3H(\dot{\Phi} + H\Phi) + \left(\frac{1}{a^2}\right)\nabla^2\Phi = (4\pi G)\delta\rho, \tag{61}$$

$$\nabla_i(\dot{\Phi} + H\Phi) = (4\pi G)(\nabla_i\delta\sigma), \tag{62}$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2)\Phi = (4\pi G)\delta p. \tag{63}$$

The first and the third of these Einstein equations can be combined to lead to the following differential equation for the Bardeen potential Φ (refs 11 and 43):

$$\begin{aligned}\Phi'' &+ 3\mathcal{H}(1 + c_{\rm A}^2)\Phi' - c_{\rm A}^2\nabla^2\Phi \\ &+ [2\mathcal{H}' + (1 + 3c_{\rm A}^2)\mathcal{H}^2]\Phi = (4\pi G a^2)\delta p^{\rm NA},\end{aligned} \tag{64}$$

where $\mathcal{H} = (a'/a)$ is the conformal Hubble parameter. In arriving at the above equation, I have changed over to the conformal time coordinate, and have made use of the standard relation[47]

$$\delta p = (c_{\rm A}^2\delta\rho + \delta p^{\rm NA}), \tag{65}$$

where $c_{\rm A}^2 \equiv (p'/\rho')$ denotes the adiabatic speed of the perturbations, and $\delta p^{\rm NA}$ represents the non-adiabatic pressure perturbation.

A conserved quantity at super-Hubble scales: Consider the following combination of the Bardeen potential Φ and its time derivative[8,11,48,49]:

$$\mathcal{R} = \Phi + \left(\frac{2\rho}{3\mathcal{H}}\right)\left(\frac{\Phi' + \mathcal{H}\Phi}{\rho + p}\right), \tag{66}$$

a quantity that is referred to as the curvature perturbation [It is called so since it is proportional to the local three curvature on the spatial hypersurface[15,19].] Upon substituting this expression in eq. (64) that describes the evolution of the potential Φ, and making use of the background equations, i.e. eqs (11) and (12), one obtains that, in Fourier space,

$$\mathcal{R}_k' = \left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right)[(4\pi G a^2)\delta p_k^{\rm NA} - c_{\rm A}^2 k^2 \Phi_k]. \tag{67}$$

(Henceforth, the subscript k shall refer to the wavenumber of the Fourier modes of the perturbations.) Note that at super-Hubble scales, wherein the physical wavelengths of the perturbations are much larger than the Hubble radius (i.e. when $(k/aH) = (k/\mathcal{H}) << 1$), the term $(c_{\rm A}^2 k^2 \Phi_k)$ can be neglected. If one further assumes that no non-adiabatic pressure perturbations are present (i.e. $\delta p^{\rm NA} = 0$), say, as in the case of ideal fluids, then the above equation implies that $\mathcal{R}_k' \simeq 0$ at super-Hubble scales. In other words, when the perturbations are adiabatic, the curvature perturbation $\mathcal{R}_k$ in conserved when the modes are well outside the Hubble radius.

Evolution of the Bardeen potential at super-Hubble scales: Let us now make use of the conservation of curvature perturbation to understand how the Bardeen potential evolves at super-Hubble scales during the radiation and matter dominated eras. If I now define[6,11,18]

$$\Phi = (\mathcal{H}/a^2\theta)\,\mathcal{U}, \tag{68}$$

where

$$\theta = \left[\frac{\mathcal{H}^2}{(\mathcal{H}^2 - \mathcal{H}')a^2}\right]^{1/2}, \tag{69}$$

then, I find that, in Fourier space, eq. (64) that governs the Bardeen potential reduces to

$$\mathcal{U}_k'' + \left[c_{\rm A}^2 k^2 - \left(\frac{\theta''}{\theta}\right)\right]\mathcal{U}_k = \left(\frac{4\pi G a^4\theta}{\mathcal{H}}\right)\delta p_k^{\rm NA}. \tag{70}$$

In the absence of non-adiabatic pressure perturbations (i.e. when $\delta p^{\rm NA} = 0$), the above differential equation for $\mathcal{U}_k$ simplifies to

$$\mathcal{U}_k'' + \left[c_{\rm A}^2 k^2 - \left(\frac{\theta''}{\theta}\right)\right]\mathcal{U}_k = 0. \tag{71}$$

And, in the super-Hubble limit, say, as $k \to 0$, the general solution to this differential equation can be written as[6,18]

$$\mathcal{U}_k(\eta) \simeq C_G(k)\theta(\eta)\int^{\eta}\frac{{\rm d}\tilde{\eta}}{\theta^2(\tilde{\eta})} + C_D(k)\theta(\eta), \tag{72}$$

where the coefficients C_G and C_D are k-dependent constants that are determined by the initial conditions imposed at early times. The corresponding Bardeen potential Φ_k is then given by

$$\Phi_k(\eta) \simeq C_G(k)\left(\frac{\mathcal{H}}{a^2(\eta)}\right)\int^{\eta}\frac{\mathrm{d}\tilde{\eta}}{\theta^2(\tilde{\eta})} + C_D(k)\left(\frac{\mathcal{H}}{a^2(\eta)}\right). \quad (73)$$

Consider the power law expansion (eq. (23)) that we had discussed earlier. Such an expansion can be expressed in terms of the conformal time coordinate as follows:

$$a(\eta) = (-\bar{\mathcal{H}}\eta)^{(\gamma+1)}, \quad (74)$$

where γ and $\bar{\mathcal{H}}$ are constants given by

$$\gamma = -\left(\frac{2q-1}{q-1}\right) \quad \text{and} \quad \bar{\mathcal{H}} = [(q-1)a_1^{1/q}]. \quad (75)$$

In addition to power law inflation (which corresponds to $q > 1$), the above scale factor describes the radiation and matter-dominated epochs (corresponding to $q = (1/2)$ and $(2/3)$ respectively) as well. During inflation, $\gamma \le -2$, with $\gamma = -2$ corresponding to exponential inflation (i.e. $q \to \infty$). While $\gamma = 0$ in the case of the radiation-dominated epoch, $\gamma = 1$ during matter-domination. Moreover, note that the quantity $\bar{\mathcal{H}}$ is positive and $-\infty < \eta < 0$ during inflation, whereas, during radiation and matter domination, $\bar{\mathcal{H}}$ is negative and $0 < \eta < \infty$. It is helpful to notice that in all these instances, $\eta \to 0$ corresponds to the super-Hubble limit.

In the background described by eq. (74), the quantity θ that is defined in eq. (69) is found to be

$$\theta(\eta) = \left(\frac{\gamma+1}{\gamma+2}\right)^{1/2}\left(\frac{1}{a(\eta)}\right), \quad (76)$$

so that, at super-Hubble scales, one has [cf. eq. (73)]

$$\Phi_k(\eta) \simeq C_G(k)\left[\frac{3(w+1)}{3w+5}\right] + C_D(k)\left[\frac{2/(3w+1)}{\bar{\mathcal{H}}^{[2(\gamma+1)]}\eta^{(2\gamma+3)}}\right], \quad (77)$$

where w is the following equation-of-state parameter:

$$w = (p/\rho) = [(1-\gamma)/3(1+\gamma)], \quad (78)$$

that is a constant in power law expansion. The first term in the above expression for Φ_k denotes the growing mode (which is actually a constant), while the second term represents the decaying mode. (Hence, the choice of subscripts G and D to the coefficient C.) Demanding finiteness at very early times implies that the decaying mode has to be neglected, so that, at super-Hubble scales, I have[18]

$$\Phi_k(\eta) \simeq C_G(k)\left[\frac{3(w+1)}{3w+5}\right]. \quad (79)$$

This quantity vanishes when $w = -1$, which corresponds to exponential expansion driven by the cosmological constant. Therefore, it is often said that the cosmological constant does not induce any metric perturbations.

Since Φ_k is a constant at super-Hubble scales, in this limit, the curvature perturbation $\mathcal{R}_k$ [cf. eq. (66)] is given by

$$\mathcal{R}_k \simeq \left[\frac{3w+5}{3(w+1)}\right]\Phi_k \simeq C_G(k), \quad (80)$$

where I have made use of eq. (79) for Φ_k. As $\mathcal{R}_k$ is conserved and Φ_k is a constant at super-Hubble scales in power law expansion, when the modes enter the Hubble radius during the radiation or matter-dominated epochs, the Bardeen potentials at entry are given by[15]

$$\Phi_k^{\mathrm{R}} \simeq \left[\frac{3(w_{\mathrm{R}}+1)}{3w_{\mathrm{R}}+5}\right]\mathcal{R}_k = \left(\frac{2}{3}\right)C_G(k), \quad (81)$$

$$\Phi_k^{\mathrm{M}} \simeq \left[\frac{3(w_{\mathrm{M}}+1)}{3w_{\mathrm{M}}+5}\right]\mathcal{R}_k = \left(\frac{3}{5}\right)C_G(k), \quad (82)$$

where I have made use of the fact that $w_{\mathrm{R}} = (1/3)$ and $w_{\mathrm{M}} = 0$. These expressions also imply that, at super-Hubble scales, Φ_k changes by a factor of (9/10) during the transition from radiation to matter-dominated epoch[5,15].

It is essentially the spectrum of the Bardeen potential when the modes enter the Hubble radius during the radiation and matter-dominated epochs that determines the pattern of anisotropies in the CMB and the structure of the universe at the largest scales that we observe today[5,7,8]. The quantity $C_G(k)$ that determines such a spectrum has to be arrived at by solving eq. (70) exactly with suitable initial conditions imposed in the early universe. In the inflationary scenario, physically well-motivated, quantum, initial conditions are imposed on the modes when they are deep inside the Hubble radius. As I shall illustrate in the following section, under these conditions, it is the background dynamics during the inflationary regime that influences the functional form of $C_G(k)$.

Vector perturbations

As I had done in the case of the scalar perturbations, I shall work in a particular gauge to describe the vector perturbations as well. I shall choose a gauge wherein the Friedmann metric, when the vector perturbations have been included, is described by the line element[6–8]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)[\delta_{ij} + (\nabla_i F_j + \nabla_j F_i)]\mathrm{d}x^i\mathrm{d}x^j. \quad (83)$$

(This corresponds to setting $\mathcal{S}_i$ to zero and choosing $\mathcal{F}_i \propto F_i$.) In such a gauge, upon using the condition that F_i is a divergence-free vector, the different components of the perturbed Einstein tensor are found to be

$$\delta G_0^0 = 0, \; \delta G_i^0 = \left(\frac{1}{2}\right)(\nabla^2 \dot{F}_i), \tag{84}$$

$$\delta G_j^i = -\left(\frac{1}{2}\right)[3H(\nabla_i \dot{F}_j + \nabla_j \dot{F}_i) + (\nabla_i \ddot{F}_j + \nabla_j \ddot{F}_i)]. \tag{85}$$

In the absence of any vector sources, according to the first-order Einstein's equations, the non-zero components δG_i^0 and δG_j^i have to be equated to zero. These then immediately imply that the metric perturbation F_i vanishes identically. In other words, no vector perturbations are generated in the absence of sources with vorticity[6,20].

Tensor perturbations

Let us now turn to the case of the tensor perturbations. Upon the inclusion of these perturbations, the Friedmann metric can be described by the line element[5–8]

$$ds^2 = dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \tag{86}$$

where h_{ij} is a symmetric, transverse and traceless tensor. (Note that $h_{ij} \propto \mathcal{H}_{ij}$.) As I had discussed, the transverse and traceless conditions reduce the number of independent degrees of freedom of h_{ij} to two. These two degrees correspond to the two types of polarization associated with the gravitational waves. It can be shown that, on imposing the transverse and the traceless conditions, the components of the perturbed Einstein tensor corresponding to the above line element simplify to[6]

$$\delta G_0^0 = \delta G_i^0 = 0, \tag{87}$$

$$\delta G_j^i = -\left(\frac{1}{2}\right)\left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - \left(\frac{1}{a^2}\right)\nabla^2 h_{ij}\right). \tag{88}$$

In the absence of anisotropic stresses, one then arrives at the following differential equation describing the amplitude h of the gravitational waves[5–8]:

$$h'' + 2\mathcal{H}h' - \nabla^2 h = 0, \tag{89}$$

where, for later use, I have expressed the equation in terms of the conformal time coordinate.

Generation of perturbations during inflation

As I have repeatedly pointed out, the striking feature of inflation is the fact that it provides a natural mechanism to generate the perturbations[50–53]. It is the quantum fluctuations associated with the inflaton that act as the primordial seeds for the inhomogeneities. I had earlier alluded to the fact that it is the spectrum of the Bardeen potential that determines the pattern of the anisotropies in the CMB and the formation of structures. Since the curvature perturbation is proportional to the Bardeen potential at super-Hubble scales (cf. eq. (80)), the primary quantity of interest is the spectrum of curvature perturbations generated during inflation. The inflaton being a scalar source, does not degenerate any vector perturbations[7,8]. However, as I have discussed, gravitational waves are generated even in the absence of sources[45,46,54]. The primordial gravitational waves are also important because they too leave their own distinct imprints on the CMB[3,5,8]. In this section, I shall first obtain the equation of motion governing the curvature perturbation when the universe is dominated by the inflaton. I shall then quantize the curvature and the tensor perturbations, impose vacuum initial conditions, and evaluate the scalar and the tensor spectra in super-Hubble limit for the cases of power law and slow-roll inflation.

Equation of motion for the curvature perturbation

As before, let ϕ denote the homogeneous scalar field. Also, let $\delta\phi$ denote the perturbation in the field. It is then straightforward to show that, in the metric (56), the components of the perturbed stress–energy tensor (cf. eqs (14) and (60)) associated with the scalar field can be expressed as

$$\delta T_0^0 = (\dot{\phi}\dot{\delta\phi} - \dot{\phi}^2\Phi + V_\phi \delta\phi) = \delta\rho, \tag{90}$$

$$\delta T_i^0 = \nabla_i(\dot{\phi}\delta\phi) = \nabla_i(\delta\sigma), \tag{91}$$

$$\delta T_j^i = -(\dot{\phi}\dot{\delta\phi} - \dot{\phi}^2\Phi - V_\phi \delta\phi)\delta_j^i = -\delta p \delta_j^i. \tag{92}$$

Evidently, the scalar field does not possess any anisotropic stress. As a result, $\Phi = \Psi$ during inflation. On substituting the above expressions for $\delta\rho$, $\delta\sigma$ and δp in the first-order Einstein equations (eqs (61)–(63)) governing the scalar perturbations, one can arrive at following equation for the Bardeen potential:

$$\Phi'' + 3\mathcal{H}(1 + c_A^2)\Phi' - c_A^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_A^2)\mathcal{H}^2]\Phi = (1 - c_A^2)\nabla^2\Phi. \tag{93}$$

Upon comparing this equation for Φ with eq. (64), it is evident that the non-adiabatic pressure perturbation associated with the inflaton is given by

$$\delta p^{\rm NA} = \left(\frac{1 - c_A^2}{4\pi G a^2}\right)\nabla^2\Phi. \tag{94}$$

In such a case, eq. (67) that describes the evolution of the curvature perturbation simplifies to

$$\mathcal{R}_k' = -\left(\frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\right)(k^2\Phi_k). \tag{95}$$

On differentiating this equation again with respect to time and, on using the background equations, i.e. eqs (19) and (20), the definition (66) and the Bardeen equation (i.e. eq. (93)), one obtains the following equation of motion governing the Fourier modes of the curvature perturbation induced by the scalar field[8,20]:

$$\mathcal{R}_k'' + 2\left(\frac{z'}{z}\right)\mathcal{R}_k' + k^2\mathcal{R}_k = 0, \tag{96}$$

where the quantity z is given by

$$z = (a\dot{\phi}/H) = (a\phi'/\mathcal{H}). \tag{97}$$

It is useful to introduce the Mukhanov–Sasaki variable v that is defined as[55,56]

$$v = (\mathcal{R}z). \tag{98}$$

The Fourier modes of this variable, say, v_k, satisfy the differential equation

$$v_k'' + \left[k^2 - \left(\frac{z''}{z}\right)\right]v_k = 0. \tag{99}$$

Quantization of perturbations and definition of the power spectra

On quantization, the homogeneity of the Friedmann background allows us to express the curvature perturbation $\mathcal{R}$ in terms of the Fourier modes $\mathcal{R}_k$ (satisfying eq. (96)) as follows[18]:

$$\hat{\mathcal{R}}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}[\hat{a}_{\mathbf{k}}\mathcal{R}_k(\eta)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger\mathcal{R}_k^*(\eta)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}]. \tag{100}$$

where the creation and the annihilation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ obey the standard commutation relations. At the linear order in the perturbation theory that I am working in, the power spectrum as well as the statistical properties of the scalar perturbations are entirely characterized by the two-point function of the quantum field $\hat{\mathcal{R}}$. (This is why the perturbations generated by inflation are often termed as Gaussian. However, basically, this happens to be true since we are restricting ourselves to the linear order in perturbation theory. I shall briefly comment on possible deviations from Gaussianity in the final section.) The power spectrum of the scalar perturbations, say, $\mathcal{P}_{\mathrm{S}}(k)$, is given by the relation

$$\int_0^\infty \frac{\mathrm{d}k}{k}\mathcal{P}_{\mathrm{S}}(k) \equiv \int \mathrm{d}^3\mathbf{k}\int\frac{\mathrm{d}^3(\mathbf{x}-\mathbf{x}')}{(2\pi)^3}\langle 0\,|\,\hat{\mathcal{R}}(\eta, \mathbf{x})\hat{\mathcal{R}}(\eta, \mathbf{x}')\,|\,0\rangle$$

$$\times \exp -\mathrm{i}[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')], \tag{101}$$

where $|0\rangle$ is the vacuum state defined as $\hat{a}_{\mathbf{k}}|0\rangle = 0\ \forall\mathbf{k}$. Using the decomposition (100), the scalar perturbation spectrum can then be obtained as[20]

$$\mathcal{P}_{\mathrm{S}}(k) = \left(\frac{k^3}{2\pi^2}\right)|\mathcal{R}_k|^2 = \left(\frac{k^3}{2\pi^2}\right)\left(\frac{|v_k|}{z}\right)^2. \tag{102}$$

The expression on the right-hand side is to be evaluated at super-Hubble scales (i.e. when $(k/aH) = (k/\mathcal{H}) << 1]$ when the curvature perturbation approaches a constant value. [Earlier, I had illustrated that, if the non-adiabatic pressure perturbation δp^{NA} can be neglected, the curvature perturbation is conserved at super-Hubble scales. It can be shown that the non-adiabatic pressure perturbation associated with the inflaton (cf. eq. (94)) decays exponentially at super-Hubble scales and, hence, can be ignored[57–59]. As a matter of fact, it is for this reason that the scalar perturbations produced by single scalar fields are usually referred to as adiabatic.]

The tensor perturbations can be quantized in a similar fashion as the curvature perturbation, and the corresponding spectrum can also be defined in the same way. If we write $h = (u/a)$, then, in Fourier space, eq. (89) describing the tensor perturbations reduces to

$$u_k'' + \left[k^2 - \left(\frac{a''}{a}\right)\right]u_k = 0. \tag{103}$$

And, it is helpful to note that this equation is essentially the same as the Mukhanov–Sasaki (eq. (99)) with the quantity z replaced by the scale factor a. The tensor perturbation spectrum, say, $\mathcal{P}_{\mathrm{T}}(k)$, can then be expressed in terms of the modes h_k and u_k as follows:

$$\mathcal{P}_{\mathrm{T}}(k) = 2\left(\frac{k^3}{2\pi^2}\right)|h_k|^2 = 2\left(\frac{k^3}{2\pi^2}\right)\left(\frac{|u_k|^2}{a}\right), \tag{104}$$

with the expressions on the right-hand side to be evaluated in the super-Hubble limit, as in the scalar case. The additional factor of two in the above tensor spectrum has been included to take into account the two states of polarization of the gravitational waves.

The scalar and the tensor spectral indices are defined as[20]

$$n_{\mathrm{s}} = 1 + \left(\frac{\mathrm{d}\ln \mathcal{P}_{\mathrm{s}}}{\mathrm{d}\ln k}\right) \text{ and } n_{\mathrm{T}} = \left(\frac{\mathrm{d}\ln \mathcal{P}_{\mathrm{T}}}{\mathrm{d}\ln k}\right). \quad (105)$$

Notice the difference in the definition of these two quantities. Conventionally, a scale-invariant scalar spectrum corresponds to $n_{\mathrm{s}} = 1$, while such a tensor spectrum is described by $n_{\mathrm{T}} = 0$. Finally, the tensor-to-scalar ratio r is defined as follows[20]:

$$r(k) \equiv \left(\frac{\mathcal{P}_{\mathrm{T}}(k)}{\mathcal{P}_{\mathrm{s}}(k)}\right). \quad (106)$$

As I shall discuss below, the scalar spectral index n_{S} and the tensor-to-scalar ratio r happen to be important inflationary parameters that can be constrained by the observations.

The Bunch–Davies initial conditions

As I have mentioned, during inflation, the initial conditions on the perturbations are imposed when the modes are well inside the Hubble scale (i.e. when $(k/aH) = (k/\mathcal{H}) >> 1$]. It is clear from eqs (99) and (103) that, in such a sub-Hubble limit, the scalar and tensor modes v_k and u_k do not feel the curvature of the spacetime and, hence, the solutions to these modes behave in the following Minkowskian form: $\mathrm{e}^{\pm(\mathrm{i}k\eta)}$. The assumption that the scalar and tensor perturbations are in the vacuum state then requires that v_k and u_k are positive frequency modes at sub-Hubble scales, i.e. they have the asymptotic form[3,6]

$$\lim_{(k/\mathcal{H})\to\infty} (v_k(\eta), u_k(\eta)) \to \left(\frac{1}{\sqrt{2k}}\right)\mathrm{e}^{-\mathrm{i}k\eta}. \quad (107)$$

It should be pointed out that the vacuum state associated with the modes that exhibit such a behaviour is often referred to in the literature as the Bunch–Davies vacuum[60].

The scalar and the tensor power spectra in power law and slow-roll inflation

In this section, my main goal will be to arrive at the scalar and tensor perturbation spectra in slow-roll inflation. However, deriving the perturbation spectra in the case of power law inflation proves to be highly instructive in understanding the computation of the slow-roll spectra. Therefore, before I derive the spectra in slow-roll inflation, I shall discuss the power law case.

The perturbation spectra in power law inflation: In power law inflation, from eqs (23) and (24), it is clear that

$$z = \left(\frac{a\dot{\phi}}{H}\right) = \sqrt{(2/q)}M_{\mathrm{P}}a. \quad (108)$$

Upon using the scale factor (74), the solution to the Mukhanov–Sasaki equation (eq. (99)) that satisfies the initial condition (107) is found to be[61–64]

$$v_k(\eta) = \left(\frac{-\pi\eta}{4}\right)^{1/2} \mathrm{e}^{\mathrm{i}[\nu+(1/2)](\pi/2)} H_\nu^{(1)}(-k\eta), \quad (109)$$

where $\nu = -[\gamma + (1/2)]$, and $H_\nu^{(1)}$ is the Hankel function of the first kind and of order ν. Since in such a power law case,

$$\left(\frac{z''}{z}\right) = \left(\frac{a''}{a}\right), \quad (110)$$

evidently, the tensor mode u_k will be described by the same solution as the one for v_k above. As a result, barring an overall constant factor, the scalar and the tensor spectra will be of the same form.

Upon using the series expansion of the Hankel function in the solution (109) and eq. (74), it can readily shown that the curvature perturbation $\mathcal{R}_k$ and the tensor amplitude h_k approach a constant value in the super-Hubble limit (i.e. as $(-k\eta) \to 0$), as expected. The scalar and the tensor power spectra evaluated at super-Hubble scales can be written as[61–64]

$$\mathcal{P}_{\mathrm{S(T)}}(k) = \mathcal{A}_{\mathrm{S(T)}}\bar{\mathcal{H}}^2 \left(\frac{k}{\bar{\mathcal{H}}}\right)^{2(\gamma+2)}. \quad (111)$$

The quantities $\mathcal{A}_{\mathrm{S}}$ and $\mathcal{A}_{\mathrm{T}}$ are given by

$$\mathcal{A}_{\mathrm{S}} = \left[\frac{\gamma+1}{16\pi^3(\gamma+2)M_{\mathrm{P}}^2}\right]\left(\frac{|\Gamma(\nu)|^2}{2^{(2\gamma+1)}}\right), \quad (112)$$

$$\mathcal{A}_{\mathrm{T}} = \left[\frac{1}{\pi^3 M_{\mathrm{P}}^2}\right]\left(\frac{|\Gamma(\nu)|^2}{2^{(2\gamma+1)}}\right), \quad (113)$$

where $\Gamma(\nu)$ is the Gamma function and, as seems to be the convention[5,20], I have included by hand, a factor of $(4/M_{\mathrm{P}}^2)$ in $\mathcal{A}_{\mathrm{T}}$. Clearly, the spectral indices are constants, and are given by

$$(n_{\mathrm{S}} - 1) = n_{\mathrm{T}} = [2(\gamma+2)] = -\left(\frac{2}{q-1}\right), \quad (114)$$

where I have made use of eq. (75). The resulting tensor-to-scalar ratio is also a constant, and is found to be

$$r = \left[\frac{16(\gamma+2)}{\gamma+1}\right] = \left(\frac{16}{q}\right). \quad (115)$$

These results point to the fact that the scalar and tensor spectra turn more and more scale-invariant as the scale factor approaches exponential inflation (i.e. as $q \to \infty$).

The power spectra in slow-roll inflation: Let us now evaluate the scalar and tensor spectra in slow-roll inflation. Using eq. (20) and eq. (36) for the first Hubble slow-roll parameter $\epsilon_{\rm H}$, the quantity z defined in eq. (97) can be written as

$$z = \sqrt{2}M_{\rm P}(a\sqrt{\epsilon_{\rm H}}). \tag{116}$$

Also, note that eqs (36) and (37) defining the two Hubble slow-roll parameters $\epsilon_{\rm H}$ and $\delta_{\rm H}$ can be expressed in terms of the conformal time coordinate as follows:

$$\epsilon_{\rm H} = 1 - \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right) \quad \text{and} \quad \delta_{\rm H} = \epsilon_{\rm H} - \left(\frac{\epsilon_{\rm H}'}{2\mathcal{H}\epsilon_{\rm H}}\right). \tag{117}$$

Using these expressions for z and the HSR parameters, the term (z''/z) that appears in the Mukhanov–Sasaki equation (eq. (99)) can be written as[13,65,66]

$$\left(\frac{z''}{z}\right) = \mathcal{H}^2\left[2 - \epsilon_{\rm H} + (\epsilon_{\rm H} - \delta_{\rm H})(3 - \delta_{\rm H}) + \left(\frac{\epsilon_{\rm H}' - \delta_{\rm H}'}{\mathcal{H}}\right)\right]. \tag{118}$$

From the definition of $\epsilon_{\rm H}$ above, it can also be established that

$$\left(\frac{a''}{a}\right) = \mathcal{H}^2(2 - \epsilon_{\rm H}). \tag{119}$$

Let us now rewrite eq. (117) above for $\epsilon_{\rm H}$ as follows:

$$\eta = -\int \left(\frac{1}{1-\epsilon_{\rm H}}\right) {\rm d}\left(\frac{1}{\mathcal{H}}\right). \tag{120}$$

On integrating this expression by parts, and using the above definition of $\delta_{\rm H}$, one obtains that

$$\eta = -\left[\frac{1}{(1-\epsilon_{\rm H})\mathcal{H}}\right] - \int \left[\frac{2\epsilon_{\rm H}(\epsilon_{\rm H} - \delta_{\rm H})}{(1-\epsilon_{\rm H})^3}\right] {\rm d}\left(\frac{1}{\mathcal{H}}\right). \tag{121}$$

At the leading order in the slow-roll approximation (cf. eq. (40)), the second term can be ignored and, at the same order, one can assume $\epsilon_{\rm H}$ to be a constant. Therefore, we have

$$\mathcal{H} \simeq -\left[\frac{1}{(1-\epsilon_{\rm H})\eta}\right]. \tag{122}$$

If we now use this expression for $\mathcal{H}$ in eqs (118) and (119), then, at the leading order in the slow-roll approximation, one gets

$$\left(\frac{z''}{z}\right) \simeq \left(\frac{2 + 6\epsilon_{\rm H} - 3\delta_{\rm H}}{\eta^2}\right), \tag{123}$$

$$\left(\frac{a''}{a}\right) \simeq \left(\frac{3 + 2\epsilon_{\rm H}}{\eta^2}\right), \tag{124}$$

with the slow-roll parameters treated as constants. It is then clear from eqs (99) and (103) that the solutions to the variables v_k and u_k will again be given in terms of Hankel functions as in the power law case, i.e. eq. (109), with the quantity ν now given by

$$\nu_{\rm S} \simeq \left[\left(\frac{3}{2}\right) + 2\epsilon_{\rm H} - \delta_{\rm H}\right] \quad \text{and} \quad \nu_{\rm T} \simeq \left[\left(\frac{3}{2}\right) + \epsilon_{\rm H}\right], \tag{125}$$

where the subscripts S and T refer to the scalar and tensor cases.

It now remains to evaluate the two spectra in the super-Hubble limit. In this limit (i.e. as $(-k\eta) \to 0$), upon expanding the Hankel function as a series about the origin, the scalar and the tensor spectra can be expressed as[20,65]

$$\mathcal{P}_{\rm S}(k) = \left(\frac{1}{32\pi^2 M_{\rm P}^2 \epsilon_{\rm H}}\right)\left[\frac{|\Gamma(\nu_{\rm S})|}{\Gamma(3/2)}\right]^2 \left(\frac{k}{a}\right)^2 \left(\frac{-k\eta}{2}\right)^{(1-2\nu_{\rm S})}$$
$$= \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \left[\frac{|\Gamma(\nu_{\rm S})|}{\Gamma(3/2)}\right]^2 2^{(2\nu_{\rm S}-3)}(1-\epsilon_{\rm H})^{(2\nu_{\rm S}-1)}, \tag{126}$$

$$\mathcal{P}_{\rm T}(k) = \left(\frac{1}{2\pi^2 M_{\rm P}^2}\right)\left[\frac{|\Gamma(\nu_{\rm T})|}{\Gamma(3/2)}\right]^2 \left(\frac{k}{a}\right)^2 \left(\frac{-k\eta}{2}\right)^{(1-2\nu_{\rm T})}$$
$$= \left(\frac{2H^2}{\pi^2 M_{\rm P}^2}\right)\left[\frac{|\Gamma(\nu_{\rm T})|}{\Gamma(3/2)}\right]^2 2^{(2\nu_{\rm T}-3)}(1-\epsilon_{\rm H})^{(2\nu_{\rm T}-1)}, \tag{127}$$

where H is the Hubble parameter, and the second equalities express the asymptotic values in terms of the values of the quantities at Hubble exit (i.e. at $(-k\eta) = (1 - \epsilon_{\rm H})^{-1}$). (Also, I have multiplied the tensor spectrum by the factor of $(4/M_{\rm P}^2)$, as I had mentioned earlier.) At the leading order in the slow-roll approximation, the amplitudes of the scalar and tensor spectra can easily be read-off from the above expressions. They are given by[20]

$$\mathcal{P}_{\rm S}(k) \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2_{k=(aH)}, \tag{128}$$

$$\mathcal{P}_{\rm T}(k) \simeq \left(\frac{8}{M_{\rm P}^2}\right)\left(\frac{H}{2\pi}\right)^2_{k=(aH)}, \tag{129}$$

with the subscripts on the right-hand side indicating that the quantities have to be evaluated when the modes cross the Hubble radius. Given a quantity, say, y, we can write[20]

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\ln k}\right)_{k=(aH)} = \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\ln a}\right)\left(\frac{\mathrm{d}\ln a}{\mathrm{d}\ln k}\right)_{k=(aH)}$$
$$\simeq \left(\frac{\dot{y}}{H}\right)_{k=(aH)}, \tag{130}$$

where, in arriving at the final expression, the following condition has been used:

$$\left(\frac{\mathrm{d}\ln a}{\mathrm{d}\ln k}\right)_{k=(aH)} \simeq 1, \tag{131}$$

as H does not vary much during slow-roll inflation. Using eqs (128) and (129) for the power spectra, the definitions of the spectral indices (eq. (105)) and eq. (130), one can easily show that

$$n_{\mathrm{S}} \simeq (1-4\epsilon_{\mathrm{H}}+2\delta_{\mathrm{H}}) \quad \text{and} \quad n_{\mathrm{T}} \simeq -(2\epsilon_{\mathrm{H}}). \tag{132}$$

These expressions unambiguously point to the fact that the scalar and tensor spectra that arise in slow-roll inflation will be nearly scale-invariant. The tensor-to-scalar ratio in the slow-roll limit is found to be

$$r \simeq (16\epsilon_{\mathrm{H}}) = -(8n_{\mathrm{T}}) \tag{133}$$

with the last equality often referred to as the consistency relation[13].

Comparison with the recent CMB observations

In this section, I shall briefly discuss how a power-law primordial spectrum compares with the recent observations of the anisotropies in the CMB. I shall also indicate how the observations constrain some of the models of inflation.

The nearly scale-invariant scalar power spectrum and the rather small tensor-to-scalar ratio that arise in slow-roll inflation seem to be in good agreement with the observations of the anisotropies in the CMB. In Figure 2, the angular power spectrum of the CMB temperature anisotropies corresponding to the concordant cosmological model – viz. a spatially flat, ΛCDM model[7], and a nearly scale-invariant primordial spectrum – has been plotted as a function of the multipoles. [The term ΛCDM model refers to the currently accepted composition of our universe, viz. about 72% of dark energy, close to 23% of cold (i.e. non-relativistic) dark matter, and roughly 5% of baryons. These numbers have been arrived at based on a variety of observations, including that of the CMB anisotropies[7,8].] Figure 2 also contains data from the most recent observations of the CMB by the WMAP mission[67]. Visually, it is clear that the concordant model provides a reasonable fit to the data. Detailed analysis of the data indicates that $n_{\mathrm{S}} \simeq 0.96$, when the tensor contribution is completely ignored. Whereas it is found that $n_{\mathrm{S}} \simeq 0.98$ when tensors are taken into account. The data also constrain the tensor-to-scalar ratio to $r < 0.43$ at 95% confidence level (CL)[68].

The slow-roll approximation also enables specific inflationary models to be compared easily with the observations. In order to do so, firstly, note that, in the slow-roll limit, upon using the corresponding background equations (i.e. eq. (41)), the scalar and tensor spectral amplitudes (i.e. eqs (128) and (129)) can be expressed in terms of the potential $V(\phi)$ and its derivative as follows[20]:

$$\mathcal{P}_{\mathrm{S}}(k) \simeq \left(\frac{1}{12\pi^2 M_{\mathrm{P}}^6}\right)\left(\frac{V^3}{V_\phi^2}\right)_{k=(aH)}, \tag{134}$$

$$\mathcal{P}_{\mathrm{T}}(k) \simeq \left(\frac{2}{3\pi^2}\right)\left(\frac{V}{M_{\mathrm{P}}^4}\right)_{k=(aH)}. \tag{135}$$

Secondly, during slow roll, the relations between the HSR and PSR parameters can be obtained to be

$$\epsilon_{\mathrm{H}} \simeq \epsilon_{\mathrm{V}} \quad \text{and} \quad \delta_{\mathrm{H}} \simeq (\eta_{\mathrm{V}} - \epsilon_{\mathrm{V}}), \tag{136}$$

and, as a result, for instance, $\epsilon_{\mathrm{V}} \simeq 1$ indicates the end of inflation. Let us now use these expressions to arrive at observational constraints on the parameters of a couple of inflationary models.

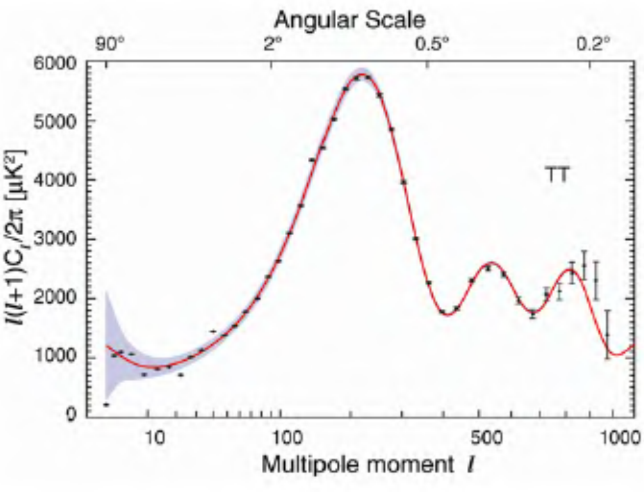

Figure 2. The angular power spectrum of CMB temperature anisotropies from the WMAP 5-year data (the black dots with error bars) and for the best-fit concordant model, i.e. a spatially flat, ΛCDM model, with a power law primordial spectrum (the solid red curve). The blue band denotes the statistical uncertainty, known as the cosmic variance[5,7,8]. The concordant model seems to fit the data rather well (from Hinshaw *et al.*[67]).

Observations indicate that the amplitude of the scalar perturbation associated with a mode that crossed the Hubble radius about 60 e-folds before the end of inflation is about (2×10^{-9}), a constraint that is referred to as the COBE normalization[69]. Given a potential, eqs (132)–(136) allow us to construct the scalar amplitude and spectral index as well as the tensor-to-scalar ratio in terms of the inflaton. Let me now focus on the large field models (eq. (31)) discussed earlier. In these models, inflation ends (i.e. $\epsilon_V \simeq 1$) when $\phi_{\mathrm{end}} \simeq [(n/\sqrt{2})M_{\mathrm{P}}]$. The value of the field at N e-folds *before the end inflaton* can be obtained using eq. (46), and is given by $\phi_{\mathrm{N}} \simeq (\sqrt{[(4N+n)n/2]}M_{\mathrm{P}})$. Using these expressions, it is straightforward to show that, for $V_0 = (m^2/2)$ and $n = 2$, the COBE normalization condition leads to the constraint that $m \simeq (10^{-5}M_{\mathrm{P}})$. Similarly, for $V_0 = (\lambda/4)$ and $n = 4$, the condition leads to $\lambda \simeq 10^{-13}$.

The CMB observations also lead to useful constraints on the models in the n_{S}–r plane. The various quantities that we have obtained above allow us to express the scalar spectral index and the tensor-to-scalar ratio in terms of the number of e-folds N *counted from the end inflaton.* For the large field models, eq. (46) enables us to arrive at the following expressions in the slow-roll limit[20]

$$n_{\mathrm{S}} \simeq 1 - \left[\frac{2(n+2)]}{4N+n}\right] \quad \text{and} \quad r \simeq \left(\frac{16n}{4N+n}\right). \tag{137}$$

I have already shown that, for the case of power law inflation, i.e. eq. (23) described by the potential in eq. (25), n_{S} and r are constants given by eqs (114) and (115) respectively. These results for the power law case were *exact*. In the slow-roll limit (i.e. when $q >> 1$), they reduce to[67]

$$n_{\mathrm{S}} \simeq 1 - \left(\frac{2}{q}\right) \quad \text{and} \quad r = \left(\frac{16}{q}\right). \tag{138}$$

These relations indicate that these models will be described by straight lines in the n_{S}–r plane. In Figure 3, the joint constraints on the n_{S}–r plane from the recent WMAP data (and a couple of other datasets) have been displayed[68]. The behaviour of a few inflationary models has been included in Figure 3 as well. It is evident from Figure 3 that, amongst the large field models, the $n = 2$ case performs better than the $n = 4$ case. Also, in the case of the power law inflation, it is found that $q < 60$ is excluded at more than 95% CL.

Status and prospects of inflation

I shall finally close by briefly commenting on the status and some prospects of the inflationary paradigm.

Profusion of inflationary models

As a broad concept, inflation can certainly be considered as a success. However, a specific model of inflation that can be satisfactorily embedded in a high-energy theory still eludes us. A plethora of inflationary models exist, quite a few of which are phenomenological in nature [I would urge the reader to take a look at the figure 41.3 in Shellard[70], which lists the various inflationary models that have been considered in the literature. The list is rather long, but, apparently, it is incomplete! It tells the tale.] Despite the enormous amount of effort, including attempts at considering models that are not described by the canonical action, it will be fair to say that we do not yet have a satisfactory model. The principal hurdle facing the idea of inflation is to construct a model that is well motivated from the high-energy perspective, and also fits the observational data well.

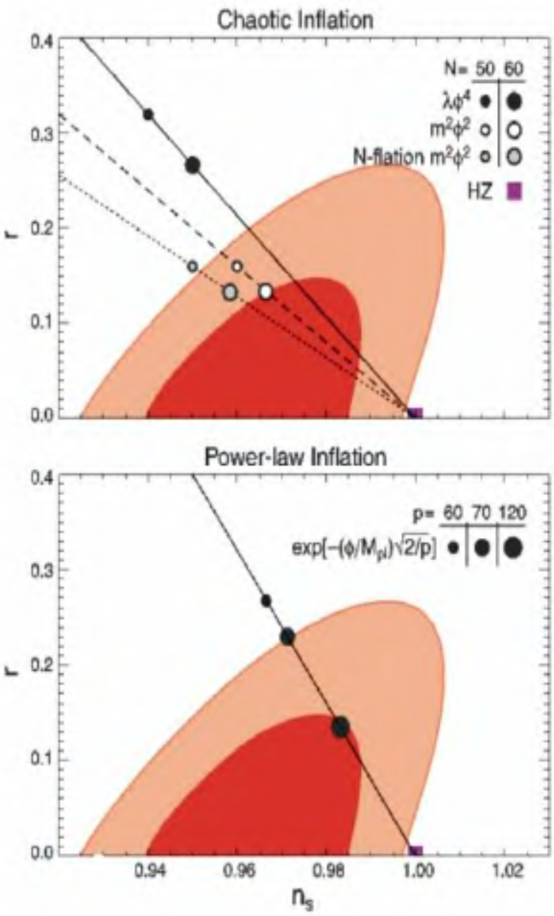

Figure 3. Joint constraints on the n_{S}–r plane from the WMAP 5-year data, and a couple of other datasets. The contours indicate the 68 and 95% CLs derived from the data. (Top panel) The data clearly exclude the $n = 4$ large field model at more than 95% CL for $N < 60$ (the solid line). In contrast, the $n = 2$ model (the dashed line) performs better, and falls at the boundary of the 68% region for $N = 60$. (I should again stress that the number of e-folds have been counted *backwards from the end of inflation*.) (Bottom panel) Behaviour of power law inflation in the slow-roll limit (i.e. when $q << 1$). It is evident that, while models with $q < 60$ are outside the 95% region, models with $q \simeq 120$ lie close to the boundary of the 68% region. (Note that p in the bottom panel is what I have called as q. I would refer the reader to Komatsu *et al.*[67] for any further details.)

Features in the primordial spectrum

In this review, I had restricted myself to discussions on slow-roll inflation, which leads to a featureless and nearly scale-invariant primordial spectrum that seems to agree well with the recent observations of the CMB anisotropies. Though the agreement is rather good, there also exist a few points at the lower multipoles of the observed CMB angular power spectrum which lie outside the cosmic variance associated with the concordant model. Given these observations, a handful of model independent approaches have been constructed to recover the primordial power spectrum. At the smaller scales, all these approaches arrive at a spectrum that is nearly scale-invariant. However, many of the approaches seem to unambiguously point to a sharp drop in power (along with a few distinct features) at the scales corresponding to the Hubble scale today. If future observations support the presence of such features in the primordial spectrum, then it poses an interesting challenge to build inflationary models that lead to the required spectrum. Conversely, the necessities of features will considerably restrict the class of allowed models of inflation. (For various efforts in this direction, see the references listed in Jain *et al.*[71,72].)

Deviations from Gaussianity

I had earlier mentioned that the scalar perturbations generated during inflation are Gaussian in nature. I had also clarified that this was primarily due to the fact that we had confined ourselves to linear perturbation theory. Deviations from Gaussianity can arise when one takes into account the perturbations at the higher orders[73,74]. However, the extent of the non-Gaussianity depends on a variety of reasons (for a recent discussion on the issue and further references, see Komatsu[75]). Interestingly, recent re-analysis of the WMAP 5-year data seems to indicate sufficiently large non-Gaussianities (see, for instance, Smith *et al.*[76]). If future observations confirm such a large level of non-Gaussianity, then it can result in a substantial tightening in the constraints on the inflationary models. For example, canonical scalar field models that lead to a nearly scale-invariant primordial spectrum contain only a small amount of non-Gaussianity and, hence, may cease to be viable[77]. However, it is known that primordial spectra with features can lead to reasonably large non-Gaussianities[78,79]. Therefore, if non-Gaussianity indeed turns out to be large, then, either one may have to reconcile with the fact that the primordial spectrum contains features, or one possibly has to seriously consider models described by non-canonical actions, some of which are known to result in large Gaussianities (see, for instance, Langlois *et al.*[80,81]).

Hopefully, one or both of these aspects will help us arrive at a satisfactory model of inflation.

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