An approximate approach for nonlinear system response determination under evolutionary stochastic excitation

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A method based on the concepts of stochastic averaging and equivalent linearization is presented for determining the response of a lightly damped nonlinear single-degree-of-freedom oscillator to a random excitation with an evolutionary broad-band power spectrum. The nonlinearities could be either of the hysteretic or the 'zero-memory' kind. Approximate analytical relationships for evaluating the response variance are derived for a number of oscillators. The efficiency and accuracy of the approach is demonstrated by pertinent digital Monte Carlo simulations. The significant advantage of the proposed approach relates to the fact that it is readily applicable for excitations possessing even non-separable evolutionary power spectra circumventing ad hoc pre-filtering and pre-processing excitation treatments associated with existing alternative schemes of linearization.

Keywords: Monte Carlo simulation, nonlinear system, stochastic process.

Introduction

A BROAD class of structural systems is subject to excitations such as seismic motions, winds and ocean waves which inherently possess the attribute of evolution in time. Therefore, to accurately predict the system behaviour under this kind of loading, realistic modelling has involved representation of these phenomena by non-stationary stochastic processes. Associated with the notion of a non-stationary stochastic process is the concept of a separable or a non-separable evolutionary power spectrum. The former relates to the evolution in time of the intensity of a process with time invariant energy–frequency relationship. The latter, which in general reflects a more realistic approach, encompasses the concept of 'local' energy distributions over frequency[1,2].

Attempts towards determining, either exactly or approximately, the response statistics of a linear oscillator under evolutionary excitation can be found in several refs 3–11. Caughey and Stumpf[3] first studied the transient response of a linear oscillator under unit step modulated white noise. The evolutionary power spectrum of the response process of an oscillator subject to a unit step modulated stationary excitation was studied in ref. 5. Explicit expressions for the second moment statistics of the response were presented in ref. 7, where the results refer to white noise excitation modulated by step, boxcar and gamma envelope functions. Moreover, approximate analytical solutions for the response amplitude statistics of a lightly damped oscillator under evolutionary excitation were derived in refs 9 and 10.

On the other hand, limited progress has been made in terms of determining the stochastic response of nonlinear systems. One of the interesting approaches of treating nonlinear oscillators under evolutionary excitation has been the coupling of the equivalent linearization method with the decomposition of the covariance matrix of the input random process[12]. In this regard, a Karhunen–Loeve spectral decomposition was used in ref. 13. It can be argued, though, that often the complexity of such approaches limits their versatility.

To address this issue, an approach is formulated in this article based on the assumed pseudo-harmonic behaviour of the response of stochastically excited and lightly damped systems. Relying on this property, an averaging scheme, first proposed by Stratonovich in the 1960s, is applied to derive a first-order stochastic differential equation for the response amplitude[14–17]. The Fokker–Planck (F–P) or forward Kolmogorov equation associated with this equation is then considered[18–20] having as stiffness and damping elements the equivalent ones obtained by a linearization scheme. Using the F–P equation with the assumption that the probability density function of the response amplitude is a time-dependent Rayleigh one, a first-order ordinary differential equation for the response variance is derived. The new approach is applied to a number of hysteretic or non-hysteretic nonlinear oscillators resulting in approximate analytical expressions for computing the time-dependent response variance. The accuracy of the proposed method is verified by pertinent Monte Carlo simulation data.

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Mathematical formulation

Determination of the equivalent linear system time-dependent elements

Consider a nonlinear single-degree-of-freedom system whose motion is governed by the differential equation

$$\ddot{x} + \beta\dot{x} + z(t, x, \dot{x}) = w(t), \tag{1}$$

where a dot over a variable denotes differentiation with respect to time t; $z(t, x, \dot{x})$ is the restoring force which could be either hysteretic or depend only on the instantaneous values of x and $\dot{x}$; β is a linear damping coefficient; and $w(t)$ represents a Gaussian, zero-mean non-stationary random process possessing an evolutionary broad-band power spectrum, $S(\omega, t)$.

Focusing on the case of a lightly damped system, it can be argued that the nonlinear oscillator (eq. (1)) exhibits a pseudo-harmonic behaviour described by the equations

$$x(t) = A(t)\cos[\omega(A)t + \phi(t)], \tag{2}$$

and

$$\dot{x}(t) = -\omega(A)A(t)\sin[\omega(A)t + \phi(t)], \tag{3}$$

in which the response amplitude envelope (A) is a slowly varying function with respect to time and, therefore, can be treated as a constant over one cycle of oscillation.

Further, adopting an equivalent linearization approach discussed in ref. 21 and described in ref. 12, a linearized counterpart of eq. (1) is

$$\ddot{x} + \beta(A)\dot{x} + \omega^2(A)x = w(t), \tag{4}$$

where the equivalent damping element and the natural frequency are assumed to be functions of the amplitude (A) of the response to partly account for the effect of the nonlinearity. Thus, defining the error between eqs (1) and (4) as

$$\varepsilon = z(t, x, \dot{x}) + [\beta - \beta(A)]\dot{x} - \omega^2(A)x, \tag{5}$$

the expressions

$$\beta(A) = \beta + \frac{\oint \dot{x}z \,\mathrm{d}t}{\oint \dot{x}^2 \,\mathrm{d}t}, \tag{6}$$

and

$$\omega^2(A) = \frac{\oint xz \,\mathrm{d}t}{\oint x^2 \,\mathrm{d}t} \tag{7}$$

are derived. This is accomplished by applying an error minimization procedure in the mean square sense, where ($\oint$) can be interpreted as 'an average over one cycle' operator. Substituting eqs (2) and (3) into eqs (6) and (7) and considering ($A(t)$) and ($\phi(t)$) constant over one cycle yields

$$\beta(A) = \beta + \frac{S(A)}{A\omega(A)}, \tag{8}$$

and

$$\omega^2(A) = \frac{C(A)}{A}, \tag{9}$$

where

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} \cos[\psi]z(t, A\cos\psi, -\omega(A)A\sin\psi)\,\mathrm{d}\psi, \tag{10}$$

and

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} \sin[\psi]z(t, A\cos\psi, -\omega(A)A\sin\psi)\,\mathrm{d}\psi. \tag{11}$$

Let the symbol $p(A, t)$ denotes the probability density function of the amplitude (A) of the response process (x). Then, the equivalent time-dependent damping factor and natural frequency can be approximated by taking expectations on the right-hand sides of eqs (8) and (9) respectively. That is,

$$\beta_{\mathrm{eq}}(t) = \beta + E\left[\frac{S(A)}{A\omega(A)}\right] = \beta + \int_0^{\infty} \frac{S(A)}{A\omega(A)} p(A,t)\,\mathrm{d}A, \tag{12}$$

and

$$\omega_{\mathrm{eq}}^2(t) = E\left[\frac{C(A)}{A}\right] = \int_0^{\infty} \frac{C(A)}{A} p(A, t)\,\mathrm{d}A. \tag{13}$$

Markovian modelling of the response envelope

Taking into account the manner by which the time-dependent natural frequency and damping factor have been determined, it can be argued that they are functions slowly varying with respect to time. Therefore, the equivalent linear system can be recast, approximately, in the form

$$\ddot{x} + \beta_{\mathrm{eq}}(t)\dot{x} + \omega_{\mathrm{eq}}^2(t)x = w(t), \tag{14}$$

where the parameters of the system of eq. (4) have been approximated by ($\beta_{\mathrm{eq}}(t)$) and ($\omega_{\mathrm{eq}}(t)$). Obviously, the amplitude (A) and phase (ϕ) can now be approximated by the equations

$$A^2(t) = x^2(t) + \left(\frac{\dot{x}(t)}{\omega_{\mathrm{eq}}(t)}\right)^2, \tag{15}$$

and

$$\phi(t) = -\omega_{\mathrm{eq}}(t)t - \tan^{-1}\left(\frac{\dot{x}(t)}{\omega_{\mathrm{eq}}(t)x(t)}\right). \tag{16}$$

Differentiating eqs (2) and (3) and taking into account eq. (14) yields

$$\dot{A}(t) = -\beta_{\mathrm{eq}}(t)A(t)\sin^2[\omega_{\mathrm{eq}}(t)t + \phi(t)] - \frac{w(t)}{\omega_{\mathrm{eq}}(t)}\sin[\omega_{\mathrm{eq}}(t)t + \phi(t)]. \tag{17}$$

Relying once more on the assumption of light damping, further simplification of eq. (17) is achieved by a combination of deterministic and stochastic averaging[9,22]. This yields

$$\dot{A}(t) = -\frac{1}{2}\beta_{\mathrm{eq}}(t)A(t) + \frac{\pi S(\omega_{\mathrm{eq}}(t), t)}{2A(t)\omega_{\mathrm{eq}}^2(t)} - \frac{(\pi S(\omega_{\mathrm{eq}}(t), t))^{1/2}}{\omega_{\mathrm{eq}}(t)}\eta(t), \tag{18}$$

which governs approximately the evolution in time of the amplitude $A(t)$. In eq. (18), $\eta(t)$ is a zero mean and delta correlated process of intensity one, i.e., $E(\eta(t)) = 0$; and $E(\eta(t)(t+\tau)) = \delta(\tau)$, with ($\delta(\tau)$) being the Dirac delta function. The importance of eq. (18) relates to the fact that it is decoupled from the phase $\phi(t)$. Thus, it is feasible to model the amplitude process $A(t)$ as a one-dimensional Markov process.

Fokker–Planck equation

The FP equation that corresponds to eq. (18) is[20,23]

$$\frac{\partial p(A,t)}{\partial t} = -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}\beta_{\mathrm{eq}}(t)A + \frac{\pi S(\omega_{\mathrm{eq}}(t), t)}{2A\omega_{\mathrm{eq}}^2(t)}\right)p(A,t)\right\} + \frac{\pi S(\omega_{\mathrm{eq}}(t), t)}{2\omega_{\mathrm{eq}}^2(t)}\frac{\partial^2 p(A,t)}{\partial A^2}. \tag{19}$$

Following a similar procedure as the one described in ref. 9, a solution of eq. (19) is attempted in the form

$$p(A,t) = \frac{A}{c(t)}\mathrm{e}^{-\frac{A^2}{2c(t)}}, \tag{20}$$

where $c(t)$ accounts for the time-dependent variance of the response process x. Substituting eq. (20) into eq. (19) and manipulating yields

$$\dot{c}(t) = -\beta_{\mathrm{eq}}(c(t))c(t) + \frac{\pi S(\omega_{\mathrm{eq}}(c(t)), t)}{\omega_{\mathrm{eq}}^2(c(t))}. \tag{21}$$

Equation (21) constitutes a first-order nonlinear ordinary differential equation for the variance of the process x, which can be solved by standard numerical schemes such as the Runge–Kutta one. Evidently, by approximating the probability density function of the non-stationary amplitude response by a time-dependent Rayleigh one, a simple expression can be derived for determining the variance of the response process.

Analytical results

Piecewise linear oscillator

The first application concerns a system with piecewise linear stiffness. Mathematically, the stiffness function can be described as

$$z(x) = \begin{cases} \omega_0^2 x, & |x| \le x_0 \\ \omega_1^2 x + x_0(\omega_0^2 - \omega_1^2)\mathrm{sign}(x), & |x| > x_0, \end{cases} \tag{22}$$

where the 'small' deflection stiffness is given by ω_0^2. When the absolute value of the displacement exceeds x_0, the stiffness changes to ω_1^2. Equivalently, making use of the Heaviside function yields

$$z(x) = \omega_0^2 x + (1 - [H(x+x_0) - H(x-x_0)]) \times (\omega_1^2 x + x_0(\omega_0^2 - \omega_1^2)\mathrm{sign}(x) - \omega_0^2 x), \tag{23}$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}. \tag{24}$$

Introducing the non-dimensional displacement $y = x/x^*$; and the non-dimensional time quantity $\tau = \omega_0 t$, eq. (23) becomes

$$z(y) = y + (1 - [H(y+1) - H(y-1)]) \times (sy + (1-s)\mathrm{sign}(y) - y), \tag{25}$$

where s is the ratio of secondary to primary elastic slope. Evaluating the integrals in eqs (8) and (9) yields

$$\beta(A) = \beta, \tag{26}$$

and

$$\omega^2(A) = \begin{cases} \dfrac{2\sqrt{1-\frac{1}{A^2}}(1-s) + A\pi s - 2A(-1+s)\csc^{-1}(A)}{\pi A}, & A > 1 \\ 1, & A < 1, \end{cases} \tag{27}$$

where

$$\csc^{-1}(A)=\sin^{-1}\left(\frac{1}{A}\right). \tag{28}$$

Using eqs (12) and (13) yields the expressions

$$\omega_{\text{eq}}^2(t) = \left(1-\mathrm{e}^{-\frac{1}{2c(t)}}+\frac{\int_0^{\infty}\left(\begin{array}{l}2\sqrt{1-\frac{1}{A^2}}(1-s)+A\pi s\\ -2A(-1+s)\csc^{-1}(A)\end{array}\right)\mathrm{e}^{-\frac{A^2}{2c(t)}}\mathrm{d}A}{\pi c(t)}\right), \tag{29}$$

and

$$\beta_{\text{eq}}(t)=\beta \tag{30}$$

for the time-dependent equivalent frequency and damping factor.

Duffing oscillator

Consider the randomly excited Duffing oscillator

$$\ddot{x}+\beta\dot{x}+\omega_0^2x+\varepsilon\omega_0^2x^3=w(t), \qquad \varepsilon>0, \tag{31}$$

for which the function $z(t,x,\dot{x})$ is defined as

$$z(x)=\omega_0^2x+\varepsilon\omega_0^2x^3. \tag{32}$$

Then, using eqs (8) and (9), the amplitude-dependent approximate equivalent natural frequency and damping term are found to be respectively,

$$\beta(A)=\beta, \tag{33}$$

and

$$\omega^2(A)=\omega_0^2\left(1+\frac{3}{4}\varepsilon A^2\right). \tag{34}$$

Substituting eqs (33) and (34) into eqs (12) and (13) respectively and taking into account eq. (20), the expressions

$$\beta_{\text{eq}}(t)=\beta, \tag{35}$$

and

$$\omega_{\text{eq}}^2(t)=\omega_0^2\left(1+\frac{3}{2}\varepsilon c(t)\right) \tag{36}$$

are obtained. Finally, the use of eqs (35), (36) and (21) leads to a first-order differential equation of the variable $c(t)$.

$$\dot{c}(t)=-\beta c(t)+\frac{\pi S\left(\sqrt{\omega_0^2\left(1+\frac{3}{2}\varepsilon c(t)\right)},t\right)}{\omega_0^2\left(1+\frac{3}{2}\varepsilon c(t)\right)}. \tag{37}$$

Bilinear oscillator

An oscillator that exhibits hysteretic behaviour of the bilinear type is next considered. Thus, the equation of motion (1) becomes

$$\ddot{y}+\beta\dot{y}+ay+(1-a)z_0=f(\tau), \tag{38}$$

where the non-dimensional displacement $y=x/x^*$ and the non-dimensional time quantity $\tau=\omega_0 t$ have been introduced; x^* is the critical value of the displacement at which the yield first occurs; ω_0 is the frequency of the oscillation corresponding to the primary elastic slope; a is the ratio of plastic to elastic stiffness, and z_0 is the hysteretic force corresponding to the elasto-plastic characteristic. The hysteretic force z_0 can be represented in terms of a first-order differential equation[24] as

$$\dot{z}_0=\dot{y}[1-H(\dot{y})H(z_0-1)-H(-\dot{y})H(-z_0-1)]. \tag{39}$$

Comparing eqs (1) and (39) yields

$$z(t)=ay+(1-a)z_0. \tag{40}$$

Using eqs (8) and (9), the amplitude-dependent equivalent elements are determined as

$$\beta(A)=\beta+\frac{(1-a)S_0(A)}{\sqrt{aA^2+(1-a)AC_0(A)}}, \tag{41}$$

and

$$\omega^2(A)=a+(1-a)\frac{C_0(A)}{A}, \tag{42}$$

where

$$C_0(A)=\frac{1}{\pi}\int_0^{2\pi}\cos[\psi]z_0(A,t)\,\mathrm{d}\psi, \tag{43}$$

and

$$S_0(A)=-\frac{1}{\pi}\int_0^{2\pi}\sin[\psi]z_0(A,t)\,\mathrm{d}\psi. \tag{44}$$

A technique for evaluating the integrals in eqs (43) and (44) can be found in refs 25 and 26, it yields

$$C_0(A)=\begin{cases}\dfrac{A}{\pi}(\Lambda-0.5\sin(2\Lambda)), & A>1\\ A, & A<1.\end{cases} \tag{45}$$

and

$$S_0(A)=\begin{cases}\dfrac{4}{\pi}\left(1-\dfrac{1}{A}\right), & A>1\\ 0, & A<1,\end{cases} \tag{46}$$

where

$$\cos 2\Lambda = 1-\frac{2}{A}. \tag{47}$$

Combining eqs (41)–(47) and (12) and (13) yields the expressions

$$\omega_{\text{eq}}^2(t)=a+(1-a)\left[1-\mathrm{e}^{-\frac{1}{2c(t)}}+\frac{1}{\pi c(t)}\int_0^\infty(\Lambda-0.5\sin 2\Lambda)A\mathrm{e}^{-\frac{A^2}{2c(t)}}\mathrm{d}A\right], \tag{48}$$

and

$$\beta_{\text{eq}}(t)=\beta+\frac{4(1-a)}{\pi c(t)}\int_0^\infty\frac{\left(1-\dfrac{1}{A}\right)}{\sqrt{a+\dfrac{(1-a)}{\pi}(\Lambda-0.5\sin 2\Lambda)}}\mathrm{e}^{-\frac{A^2}{2c(t)}}\mathrm{d}A \tag{49}$$

for the time-dependent equivalent frequency and damping factor.

Preisach oscillator

Recently, an envelope-based approach has been applied in ref. 27 to determine the response amplitude statistics of Preisach hysteretic systems under stationary Gaussian white noise excitation. The approach has been further extended in ref. 28 to yield response energy envelope statistics. Following the notation introduced in ref. 27, the equation of motion (1) becomes

$$\ddot{x}+\beta\dot{x}+\bar{\omega}^2x+f_H(t)=w(t), \tag{50}$$

where

$$\bar{\omega}=\sqrt{\omega_0^2+\omega_j^2}=\omega_j\sqrt{1+\phi}, \tag{51}$$

$$\omega_j=\sqrt{k_j}, \tag{52}$$

and

$$\phi=\omega_0^2/\omega_j^2, \tag{53}$$

As mentioned in ref. 27, the Preisach restoring force can be divided into two terms; a linear part (ω_j^2x) and a nonlinear one ($f_H(t)$) monitoring the memory of the system. Therefore, ϕ quantifies the stiffness of the linear counterpart of the Preisach element compared to the linear stiffness ω_0^2 contribution. Introducing now the parameter

$$\psi=\frac{\bar{\omega}^2}{f_y^*}, \tag{54}$$

eq. (50) can be recast in the form

$$\ddot{x}+\beta\dot{x}+\bar{\omega}^2(x+\psi d_H(t))=w(t), \tag{55}$$

where ($d_H(t)$) is the scaled hysteretic restoring force. Further,

$$f_y^*=\frac{f_{y,\max}+f_{y,\min}}{2}, \tag{56}$$

where f_y is the yielding force. Defining the non-dimensional parameter ν as

$$\nu=\frac{f_{y,\max}-f_{y,\min}}{2f_y^*}, \tag{57}$$

and applying eqs (8) and (9) for $\nu = 1$ yields

$$\beta(A)=\beta+\frac{\psi\bar{\omega}^2}{3\pi(1+\phi)^2\sqrt{\bar{\omega}^2-\dfrac{\psi\bar{\omega}^2}{4(1+\phi)^2}A}}A, \tag{58}$$

and

$$\omega^2(A)=\bar{\omega}^2-\frac{\psi\bar{\omega}^2}{4(1+\phi)^2}A. \tag{59}$$

Equivalent expressions can be found for arbitrary values of ν, though more complicated. Combining eqs (58) and (59), and (12) and (13) leads to the expressions

$$\omega_{\text{eq}}^2(t)=\bar{\omega}^2\left(1-\frac{\psi\sqrt{2\pi c(t)}}{8(1+\phi)^2}\right), \tag{60}$$

and

$$\beta_{\mathrm{eq}}(t) = \beta + \frac{\psi\varpi^2}{3\pi(1+\phi)^2 c(t)} \times \int_0^\infty \frac{A^2}{\sqrt{\varpi^2 - \dfrac{\psi\varpi^2}{4(1+\phi)^2}A}} \mathrm{e}^{-\frac{A^2}{2c(t)}} \mathrm{d}A \tag{61}$$

for the time-dependent equivalent elements.

Numerical applications

To assess the accuracy of the proposed method, digital simulations have been performed considering both separable and non-separable excitations. For each Monte Carlo simulation an ensemble size of 500 realizations has been used, whereas the value of 0.01 has been chosen for the ratio of critical damping (ζ).

Separable processes

In the case of a separable random process, the evolutionary power spectrum of the excitation is taken in the form

$$S(\omega, t) = |g(t)|^2 S_V(\omega), \tag{62}$$

where $g(t)$ is a slowly varying time-dependent modulating function; $S_V(\omega)$ is the power spectrum of a stationary process $v(t)$. In the particular case considered herein, the excitation process is recast in the form

$$w(t) = g(t)v(t), \tag{63}$$

with the modulating function given by the equation

$$g(t) = k(\mathrm{e}^{-at} - \mathrm{e}^{-bt}), \tag{64}$$

in which $a = 0.25$, $b = 0.5$, k is a normalization constant so that $g_{\max} = 1$.

Modulated Gaussian white noise: The case where $S_V(\omega) = S_0$, $0 \leq |\omega| \leq \infty$ is first considered. Obviously, for the case of a modulated white noise excitation, there exist several approaches for evaluating the response statistics. However, this simulation serves the purpose of comparing the proposed approach to another equivalent linearization scheme. The latter, equally simple to implement for modulated white noise, is generally expected to have greater accuracy, since it does not have the element of averaging. Extended presentation of the alternative scheme exists in ref. 12; therefore, limited background information is included herein.

The simulation study is restricted to the case of a Duffing oscillator. Based on the assumption of Gaussian approximation of the response, eqs (35) and (36) yield

$$\beta_{\mathrm{eq}}(t) = \beta, \tag{65}$$

and

$$\omega_{\mathrm{eq}}^2(t) = \omega_0^2(1 + 3\varepsilon c(t)) \tag{66}$$

for the non-stationary linearization method. The variance of the response is then determined by solving the set of coupled differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}E(x^2) = 2E(x\dot{x}),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(x\dot{x}) = -(\omega_0^2(1+3\varepsilon E(x^2)))E(x^2) - \beta E(x\dot{x}) + E(\dot{x}^2),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\dot{x}^2) = 2\pi|g(t)|^2 S_0 - 2(\omega_0^2(1+3\varepsilon E(x^2)))E(x\dot{x}) - 2\beta E(\dot{x}^2). \tag{67}$$

The equivalent to the set of eqs (67) for the proposed approach is eq. (21), which herein becomes

$$\dot{c}(t) = -\beta c(t) + \frac{\pi|g(t)|^2 S_0}{\omega_0^2\left(1 + \frac{3}{2}\varepsilon c(t)\right)}. \tag{68}$$

The results obtained by eqs (67) and (68), along with the digital data, are shown in Figures 1 and 2. For the natural frequency (ω_0), the value 3.61 rad/s has been used, whereas the values $\varepsilon = 0.5$ and 1 have been considered in Figures 1 and 2 respectively. For small values of the power spectrum (S_0), it is seen that both methods are in excellent agreement with the Monte Carlo data. Further-

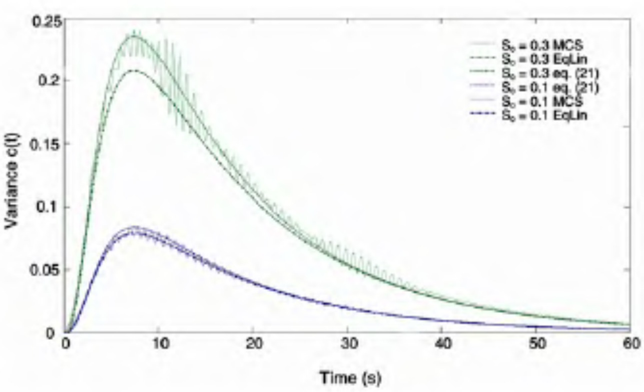

Figure 1. Response variance for a Duffing oscillator ($\varepsilon = 0.5$) under modulated Gaussian white noise. Comparison between MCS data (500 realizations), eqs (67) and (68).

more, it is shown that increasing the nonlinearity degree gradually results in divergence from the digital data as expected. However, the behaviour of the new approach indicates at least the same reliability level as the equivalent linearization one.

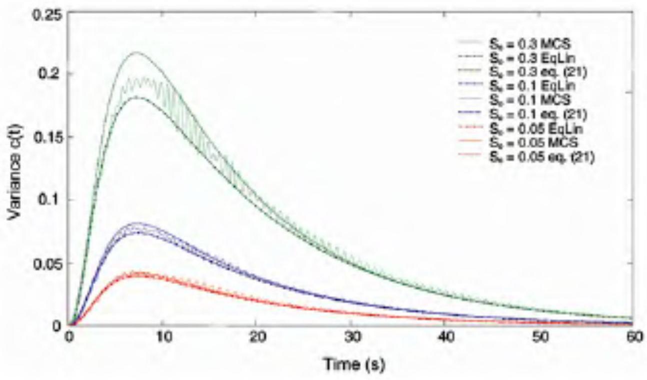

Figure 2. Response variance for a Duffing oscillator ($\varepsilon = 1$) under modulated Gaussian white noise. Comparison between MCS data (500 realizations), eqs (67) and (68).

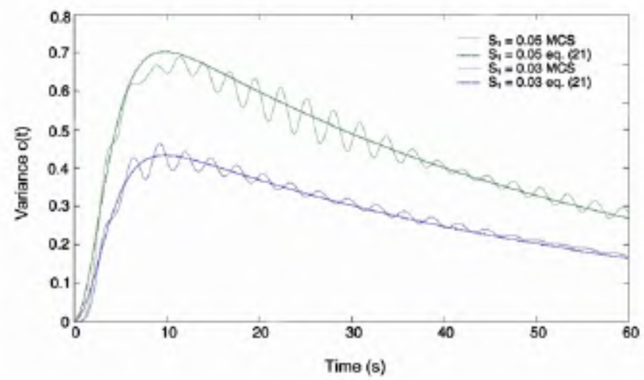

Figure 3. Response variance for an oscillator with piecewise linear stiffness ($s = 2$) under modulated Kanai–Tajimi spectrum. Comparison between MCS data (500 realizations) and eq. (21).

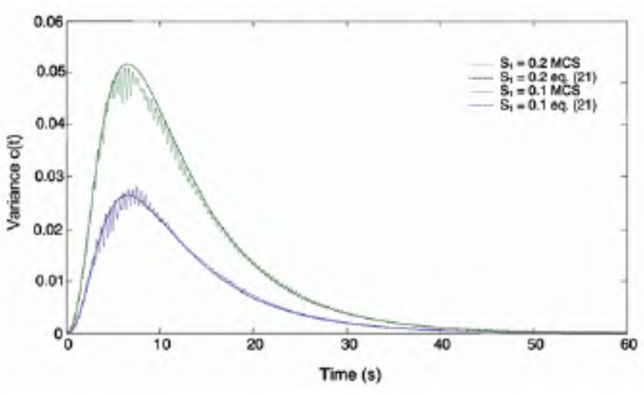

Figure 4. Response variance for a Duffing oscillator ($\varepsilon = 1$) under modulated Kanai–Tajimi spectrum. Comparison between MCS data (500 realizations) and eq. (21).

Modulated Kanai–Tajimi spectrum: The modulated Kanai–Tajimi[29,30] excitation has been frequently used in earthquake engineering applications. The following form for the power spectrum is considered

$$S_v(\omega) = S_1 \frac{(8\pi)^4 + 4(0.8)^2(8\pi)^2\omega^2}{((8\pi)^2 - \omega^2)^2 + 4(0.8)^2(8\pi)^2\omega^2},$$

$$-\infty < \omega < \infty. \qquad (69)$$

This relates to the squared modulus of the frequency response function of a single-degree-of-freedom oscillator with prescribed stiffness and damping elements. Generating realizations of the process $v(t)$ compatible with $(S_v(\omega))$ is possible by using an auto-regressive time series algorithm[31]. Specifically, values of the $(v(t))$ at equally spaced points along the time axis have been generated by using the equation

$$v_n = b_0 s_n - \sum_{k=1}^{p} b_k v_{n-k}, \qquad (70)$$

where ($v_n = v(nT)$, $T = \pi/\omega_c$) and ω_c is a selected upper cut-off frequency of the portion of $S_v(\omega)$ sought to be reproduced. Further, $s_n = s(nT)$ represents a white noise with two-sided power spectrum equal to one. A minimization procedure results in a Toeplitz system of linear equations known as the Yule–Walker equations. Solving these equations leads to the determination of the parameters (b_k). In Figures 3–5, the time evolution of the response variance under modulated Kanai–Tajimi excitation is plotted. In Figure 3, an oscillator possessing a piecewise linear stiffness is concerned. The value ($s = 2$) is used. In Figures 4 and 5, a Duffing ($\varepsilon = 1$) and a hysteretic bilinear one ($a = 0.02$, $b = 0.1$), are considered respectively. Reliable performance of the new approach is seen for various values of the input strength (S_1). Comparing the new approach results to Monte Carlo

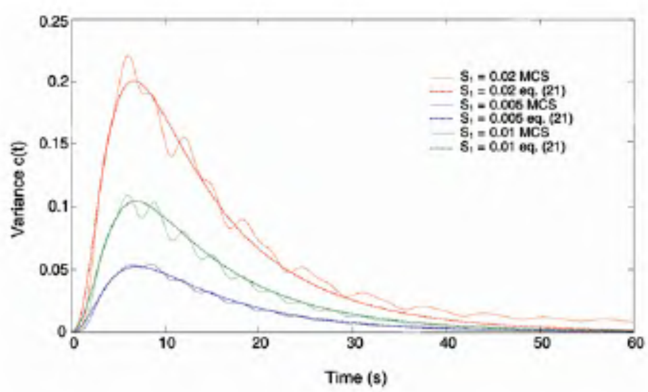

Figure 5. Response variance for a bilinear oscillator ($a = 0.02$, $b = 0.1$) under modulated Kanai–Tajimi spectrum. Comparison between MCS data (500 realizations) and eq. (21).

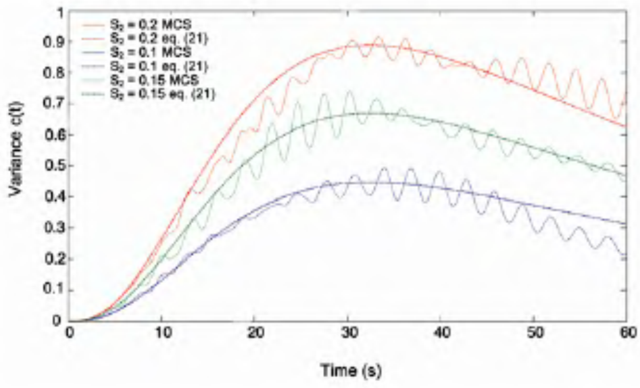

Figure 6. Response variance for an oscillator with piecewise linear stiffness ($s = 2$) under non-separable excitation. Comparison between MCS data (500 realizations) and eq. (21).

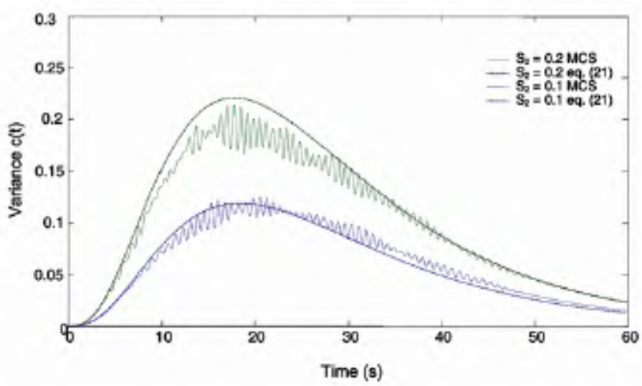

Figure 7. Response variance for a Duffing oscillator ($\varepsilon = 1$) under non-separable excitation. Comparison between MCS data (500 realizations) and eq. (21).

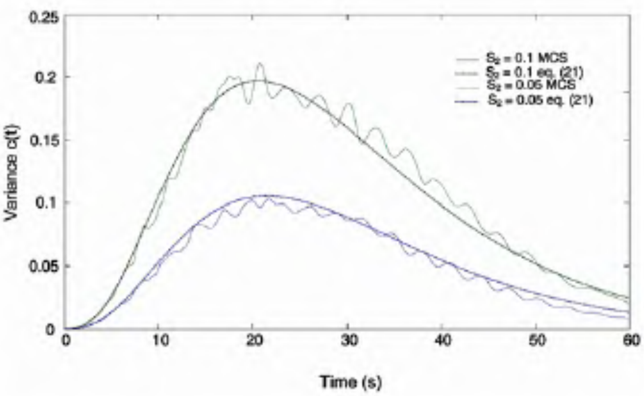

Figure 8. Response variance for a bilinear oscillator ($a = 0.02$, $b = 0.1$) under non-separable excitation. Comparison between MCS data (500 realizations) and eq. (21).

data, it can be argued that it captures successfully the time evolution of the mean value of the variance, which is quite predictable taking into account the averaging procedure which is involved.

Non-separable processes

The non-separable power spectrum

$$S(\omega, t) = S_2 \left(\frac{\omega}{5\pi}\right)^2 \mathrm{e}^{-0.15t} t^2 \mathrm{e}^{-\left(\frac{\omega}{5\pi}\right)^2 t}, \quad t \geq 0, \quad -\infty < \omega < \infty \tag{71}$$

is considered next. This spectrum comprises some of the main characteristics of seismic shaking, such as decreasing of the dominant frequency with time. Realization records compatible with eq. (71) have been produced using the concept of spectral representation of a stochastic process[31,32]. Specifically, the equation

$$w(t) = \sum_{k=1}^{m} \sqrt{4S(k\Delta\omega, t)\Delta\omega} \cos((k\Delta\omega)t + \theta_k) \tag{72}$$

has been used, in which $\Delta\omega = \omega_c/m$; and θ_k is a phase angle uniformly distributed over the interval $(0, 2\pi)$. In Figures 6–8, the time evolution of the response variance corresponding to the non-separable excitation process is plotted. Several values for the excitation level (S_2) are considered. Specifically, in Figure 6, an oscillator possessing a piecewise linear stiffness is concerned. The value ($s = 2$) has been used. In Figures 7 and 8, Duffing oscillator ($\varepsilon = 1$) and hysteretic bilinear one ($a = 0.02$, $b = 0.1$) respectively are examined. Again, it is seen that the new approach succeeds in capturing the average characteristics of the variance, while neglecting the oscillatory components.

Concluding remarks

In this article, the non-stationary response of nonlinear oscillators under evolutionary excitation has been studied. A new approach has been proposed which comprises the elements of stochastic averaging[33,34] and statistical linearization[35,36]. Specifically, taking into account the equivalent time-dependent frequency and damping factor, a simple first-order ordinary differential equation has been derived for the response variance. For this purpose, a time-dependent Rayleigh distribution for the response amplitude has been assumed. Analytical expressions have been derived for a number of hysteretic and non-hysteretic nonlinear oscillators.

Extensive digital studies have demonstrated the capacity of the approach to capture successfully the time evolution of the response variance. Indeed, the new approach succeeds in capturing the temporal average characteristics of the variance, while neglecting the oscillatory components. It appears that the proposed approach performs well for a broad class of nonlinear, elastic and inelastic oscillators. It affords the option of treating problems which involve non-separable and non-white excitation spectra without resorting to ad hoc pre-filtering or other spectral manipulation of the system excitation as is the case for many of the existing linearization schemes[12]. Furthermore, based on the demonstrated reasonable reliability of the proposed approach for determining the nonlinear response variance, it can be argued that the evolving Rayleigh distribution given by eq. (20) can be used as a logical approximation of the system response non-stationary probability density function.

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